DEPARTMENT OF MATHEMATICS

PhD IN MATHEMATICS
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C^*-dynamical systems and Ergodic Theory

Quantum decoherence, twisted tensor products and De Finetti theorem

Advisor
Prof. Francesco Fidaleo

Candidate
Elia Vincenzi

Coordinator
Prof. Carlangelo Liverani

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Chapter I

Quantum decoherence for Markov chains

I.1. Introduction

In the general framework of Quantum Mechanics, the *decoherence* phenomenon is related to the interaction of a quantum system with the surrounding environment, universally accepted as the mechanism responsible for the emergence of *classicality* in quantum dynamics and giving a dynamical explanation to the collapse of the wave function caused by a measurement procedure (for standard physical references, see for instances [83], [48] and the monograph [93]). More recently, N. P. Landsman (p. 419-420 in [112]) asserts

> *Decoherence theorists have made the point that "measurement" is not only a procedure carried out by experimental physicists in their labs, but takes place in Nature all the time without any human intervention.*

and further on

> *Which ideas have solved the problem of explaining the appearance of the classical world from quantum theory? In our opinion, none have, although since the founding days of quantum mechanics a number of new ideas have been proposed that almost certainly will play a role in the eventual resolution, should it ever be found. These ideas surely include:*
>
> - *the limit $\hbar \to 0^+$ of small Planck's constant (coming of age with the mathematical field of microlocal analysis);*
> - *the limit $N \to +\infty$ of a large system with N degrees of freedom (studied in a serious way only after the emergence of C^*-algebraic methods);*
> - *decoherence and consistent histories.*

Intuitively, it takes place when the reduced density matrices of the system diagonalize w.r.t. a particular vector basis, selected by the interaction. In this way, the phase relations between super-positions of certain vectors of the Hilbert space associated to the system are destroyed, and effects of quantum interference become essentially undetectable. In quantum optics and quantum computation, it results desirable to minimize the impact of decoherence phenomenon, for instance by selecting sectors (in the space of states) undergoing unitary evolution, and thus surviving dissipative effects. This idea, amongst others, gives rise to the mathematical axiomatization of quantum decoherence introduced by P. Blanchard and R. Olkiewicz (see [110] and [9]): the key-point is that, when decoherence takes place, the algebra describing the system can be split into a maximal subalgebra, named *decoherence-free* sector, upon which the evolution is reversible (i.e. conservative, Hamiltonian), and a complementary subspace on which the dynamics vanishes in time. Roughly speaking, after a sufficiently long time, the system then behaves *as if* it was isolated, and the maximal subalgebra turns out to be the effective algebra of observables after decoherence. We refer to [9], [60], [61], [14] and [27] for other considerations on the phenomenon, motivations for this mathematical approach and a more precise description of particular cases and consequences, such as *super-selection*.

Now, our interest in decoherence focuses on the purely mathematical viewpoint of the setting, precisely on the properties of decomposition into the persistent and transient parts of a dissipative C^*-dynamical system, typically representing a "small" system interacting with a huge reservoir. Such kind of C^*-dynamical systems are usually described by a strongly continuous one-parameter semigroup. However, in order to capture most of the main properties, we can still consider discrete dynamics generated by completely positive unital (c.p.u., for short) linear maps. In all the forthcoming analysis, we restrict the matter to this simpler picture. Let $(\mathfrak{A}, \phi)$ be a C^*-dynamical system consisting of a unital C^*-algebra $\mathfrak{A}$ on which the c.p.u. map ϕ is acting. In this specific context, the usual definition of *decoherence* requires that the algebra $\mathfrak{A}$ can be split into a direct sum as $\mathfrak{A} = M_\phi \oplus \mathfrak{A}_{\mathrm{tr}}$, where

$$M_\phi := \{x \in \mathfrak{A} \colon \phi(x^*x) = \phi(x^*)\phi(x),\, \phi(xx^*) = \phi(x)\phi(x^*)\}\,,$$

is the *multiplicative domain* of ϕ, and

$$\mathfrak{A}_{\mathrm{tr}} := \left\{x \in \mathfrak{A} \colon \lim_{n\to+\infty} \|\phi^n x\| = 0\right\}$$

is the *ϕ-transient part*. One can easily see that M_ϕ is a (possibly, not ϕ-stable) unital C^*-subalgebra of $\mathfrak{A}$ and that $\phi|_{M_\phi}$ is a *-homomorphism. Moreover, $\mathfrak{A}$ is a M_ϕ-bimodule (by defining left and right M_ϕ-module actions on $\mathfrak{A}$ respectively as $x.y := \phi(x)y$ and $y.x = y\phi(x)$, with $x \in M_\phi$, $y \in \mathfrak{A}$) and ϕ is a M_ϕ-bimodule map (see Theorem 3.18 and the discussion below in [96], p. 38-39). We recall that ϕ is *gapped* if its peripheral spectrum $\sigma_{\mathrm{per}}(\phi) := \sigma(\phi) \cap \mathbb{T}$ is topologically separated by the remainder $\sigma(\phi) \setminus \sigma_{\mathrm{per}}(\phi)$, namely there exists $r \in (0,1)$ for which $\sigma(\phi) \setminus \sigma_{\mathrm{per}}(\phi) \subset \overline{B_r(0)} \subset B_1(0)$, or equivalently $d\left(\sigma_{\mathrm{per}}(\phi), \sigma(\phi) \setminus \sigma_{\mathrm{per}}(\phi)\right) > 0$. This requirement allows to perform holomorphic functional calculus to get the *Riesz projection* relative to $\sigma(\phi) \setminus \sigma_{\mathrm{per}}(\phi)$:

$$Q_\phi := \frac{1}{2\pi i} \oint_\gamma (zI_{\mathfrak{A}} - \phi)^{-1} \mathrm{d}z \quad \in \mathcal{B}(\mathfrak{A})$$

where the contour $\gamma \colon \mathcal{I} \to B_1(0)$ is a (counterclockwise oriented, rectifiable) Jordan curve surrounding $\sigma(\phi) \setminus \sigma_{\mathrm{per}}(\phi)$ (observe that the integral is perfectly well defined, being $\gamma(\mathcal{I}) \subset \rho(\phi)$). It can easily be shown that

(1) Q_ϕ is idempotent (i.e. $Q_\phi^2 = Q_\phi$)

(2) once set $P_\phi := I_{\mathfrak{A}} - Q_\phi$, $\mathfrak{A}$ decomposes into a topological direct sum of the form

$$\mathfrak{A} = P_\phi(\mathfrak{A}) \oplus Q_\phi(\mathfrak{A})$$

where $P_\phi(\mathfrak{A})$ is a norm-closed *operator system* (i.e. a *-closed subspace containing $\mathbb{1}_{\mathfrak{A}}$), whereas $Q_\phi(\mathfrak{A})$ is just a norm-closed *operator space* (i.e. merely a subspace) in $\mathfrak{A}$

(3) $[Q_\phi, \phi] = [P_\phi, \phi] = 0$, whence both $P_\phi(\mathfrak{A})$ and $Q_\phi(\mathfrak{A})$ are ϕ-stable

(4) $\{\phi^n Q_\phi\}_{n\in\mathbb{N}} \subset \mathcal{B}(\mathfrak{A})$ uniformly converges to the zero operator, and hence

$$Q_\phi(\mathfrak{A}) = \left\{x \in \mathfrak{A} \colon \lim_{n\to+\infty} \|\phi^n(x)\| = 0\right\} = \mathfrak{A}_{\mathrm{tr}}$$

(see Proposition 3.1 in [34], p. 110).

In contrast to M_ϕ, the *peripheral space* $P_\phi(\mathfrak{A})$ may abruptely fail to be closed under the product of $\mathfrak{A}$, so not immediately inheriting a structure of C^*-subalgebra. Since $P_\phi(\mathfrak{A})$ corresponds to the persistent part of the direct sum decomposition of $\mathfrak{A}$ above, it results natural to ask if we can hope to endow $P_\phi(\mathfrak{A})$ with a well defined product, making it a C^*-algebra. It turns out that all depends on the properties of P_ϕ. In general $\|Q_\phi\| \leq R \max_{t\in[0,2\pi]} \|(Re^{it}I_{\mathfrak{A}} - \phi)^{-1}\|$ where $\mathrm{diam}(\sigma(\phi) \setminus \sigma_{\mathrm{per}}(\phi)) < R < 1$ and $1 \leq \|P_\phi\|$, thus it is not even guaranteed that P_ϕ is a contraction (this happens exactly when it is positive, i.e. when $Q_\phi(x^*x) \leq x^*x$, $x \in \mathfrak{A}$). Nonetheless, if P_ϕ is completely positive, the operator system $P_\phi(\mathfrak{A})$ inherits a structure of C^*-algebra with unit $\mathbb{1}_{\mathfrak{A}}$, when endowed with the bilinear, associative product

$$x \circ_\phi y := P_\phi(xy), \quad x, y \in P_\phi(\mathfrak{A})$$

and the restriction of the C^*-norm of $\mathfrak{A}$. This construction is contained in the proof of a theorem by Choi and Effros concerning injective operator systems (see Theorem 3.1 in [16]), and $\circ_\phi$ is then called *Choi-Effros product* associated to ϕ. Luckily, P_ϕ is completely positive at least for all finite-dimensional C^*-algebras, namely finite direct sums of full matrix algebras over $\mathbb{C}$ (see Theorem 2.1 at p. 1467 in [54] and Proposition 5.3 at p. 119 in [34]). In such a case, all self-maps are evidently gapped ($\sigma(\phi) = \sigma_{\mathrm{p}}(\phi)$ is finite) and complete positivity is equivalent to $\dim_{\mathbb{C}}(\mathfrak{A})$-positivity, thanks to the Choi theorem (see [15]).

At this stage, in the finite-dimensional setting, $\mathfrak{A}_\phi := (P_\phi(\mathfrak{A}), \circ_\phi)$ is a full-fledged unital C^*-algebra but it is still not clear if $\phi|_{\mathfrak{A}_\phi} \in \mathsf{Aut}(\mathfrak{A}_\phi)$, thus providing a conservative C^*-system $(\mathfrak{A}_\phi, \mathbb{Z}, \phi|_{\mathfrak{A}_\phi})$ where the additive group $\mathbb{Z}$ acts on $\mathfrak{A}_\phi$ via integers powers of $\phi|_{\mathfrak{A}_\phi}$. More generally, one might formulate the following speculation, as done in Conjecture 5.5 of [34] (p. 120):

Let ϕ be a c.p.u. self-map on a unital C^*-algebra $\mathfrak{A}$ s.t. $\sigma(\phi) \subset \mathbb{T}$. Then, $\phi \in \mathsf{Aut}(\mathfrak{A})$.

The essential purpose of this brief chapter is to prove the conjecture true when $\mathfrak{A}$ is also abelian, thus isomorphic to $\mathbb{C}^n$ for some $n \in \mathbb{N}$, when ϕ boils down to be a *stochastic matrix* describing the dynamics of a (finite-dimensional) Markov chain (see [100]). In other words, any finite-dimensional Markov chain encodes a genuine reversible dynamical system, after separating the persistent part from the transient one, the latter vanishing in the limit taken on the iterations of the acting stochastic matrix. In the concrete situation, up to measurement errors, this happens after a finite number of iterations, depending on the size of the so-called *mass gap*, i.e. the distance between the peripheral spectrum and the part lying inside the unit disk.[1] This is the content of the paper [36] by Fidaleo F. and Vincenzi E. (September 2022). A year after the publication of our work, in September 2023 the conjecture has been proved to be true in [8] (Theorem 3.1 at p. 201 and Remark 3.5 at p. 203) for *any* finite-dimensional C^*-algebra, thus feeding our suspicion of its validity in a more general setting. But this is not the case. Thanks to the collaboration of Glück J., we have recently been able to show that the conjecture is no longer generally true in the ∞-dimensional case, even if $\mathfrak{A}$ is abelian: there exists a (relatively easy) example of (completely) positive, unital self-map $\phi \colon \mathcal{C}(X) \to \mathcal{C}(X)$, where X is an infinite set of points, s.t. $\sigma(\phi) \subset \mathbb{T}$ but $\phi \notin \mathsf{Aut}(\mathcal{C}(X))$. After useful preliminary facts in Section I.2, we report some well-known results about c.p.u. maps on finite-dimensional

[1] The term "mass gap" comes from physical motivations, being dimensionally equivalent to a mass.

C^*-algebras in Section I.3. We then focus on (completely) positive, unital maps on $\mathbb{C}^n$, namely *stochastic* square matrices modelling finite-dimensional Markov chains (Section I.4). Using simple techniques from linear algebra, we show that any Markov chain encodes a conservative C^*-system on the persistent portion, corresponding to the peripheral eigenspace, the linear subspace generated by the eigenvectors pertaining to the peripheral eigenvalues of the stochastic matrix. In Section I.5, we mention a result by [8] which generalizes our approach to every finite-dimensional, unital C^*-algebra and expose an illustrative counterexample in the ∞-dimensional setting. We conclude this chapter with some Physics-related considerations in Section I.6.

I.2. Preliminaries

I.2.1 Basic notation

All involved C^*-algebras $\mathfrak{A}$ will have unit $\mathbb{1}_{\mathfrak{A}}$. For the unital Banach algebra $\mathcal{B}(\mathfrak{A})$ consisting of all bounded operators acting on the C^*-algebra $\mathfrak{A}$, we also put $I_{\mathfrak{A}} := \mathbb{1}_{\mathcal{B}(\mathfrak{A})}$. For involutive algebras $\mathfrak{C}_i$, $i = 1, 2$, a map $\Psi : \mathfrak{C}_1 \to \mathfrak{C}_2$ is said to be *selfadjoint* (or *real*) if $\Psi(x^*) = \Psi(x)^*$ for every $x \in \mathfrak{C}_1$. For the C^*-algebra $\mathfrak{A}$, the map $\phi : \mathfrak{A} \to \mathfrak{A}$ is said to be *completely positive* if $\phi \otimes I_{M_n(\mathbb{C})} =: \phi_n : M_n(\mathfrak{A}) \to M_n(\mathfrak{A})$ is positive for each $n \geq 1$. In particular, it is positive if ϕ_1 is. It is *unital* if $\phi(\mathbb{1}_{\mathfrak{A}}) = \mathbb{1}_{\mathfrak{A}}$. If $\mathfrak{A}$ is abelian, then complete positivity coincides with positivity (see Theorems 1.2.4 and 1.2.5 in [102], p. 3-4). For a self-map $T \in \mathcal{B}(\mathfrak{X})$ on a Banach space $\mathfrak{X}$, the *peripheral spectrum* is defined as $\sigma_{\text{per}}(T) := \{\lambda \in \sigma(T) \colon |\lambda| = r(T)\}$, $r(T) := \sup\limits_{\sigma(T)} |\lambda|$ being the *spectral radius* of T. Such a map T is said to be *gapped* if it presents the so-called *mass-gap*, that is $d\left(\sigma_{\text{per}}(T), \sigma(T) \setminus \sigma_{\text{per}}(T)\right) > 0$. In the case of a positive unital map ϕ on a unital C^*-algebra $\mathfrak{A}$, $1 \leq r(\phi) \leq \|\phi\| = \|\phi(\mathbb{1}_{\mathfrak{A}})\| = 1$, whence $r(\phi) = \|\phi\| = 1$ and $\sigma_{\text{per}}(T)$ is contained in the unit circle $\mathbb{T} := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$. Moreover, if the c.p.u. map ϕ is gapped, then $\mathfrak{A} = P_\phi(\mathfrak{A}) \oplus Q_\phi(\mathfrak{A})$. Here,

$$Q_\phi := \frac{1}{2\pi i} \oint\limits_{\gamma} (z I_{\mathfrak{A}} - \phi)^{-1} \mathrm{d}z \quad \in \mathcal{B}(\mathfrak{A})$$

where the contour $\gamma \colon \mathcal{I} \to B_1(0)$ is a counterclockwise oriented, rectifiable Jordan curve surrounding $\sigma(\phi) \setminus \sigma_{\text{per}}(\phi)$, whereas $P_\phi := I - Q_\phi$. By Proposition 3.1 in [34] (p. 110), $\lim\limits_{n \to +\infty} \|\phi^n(x)\| = 0$, $x \in Q_\phi(\mathfrak{A})$.

I.2.2 Order isomorphisms

Any (complex, associative) algebra A has a *Jordan algebra* structure as well, if endowed with the Jordan product

$$A \ni a, b \mapsto a \bullet b := \frac{1}{2}(ab + ba) \in A$$

which coincides with the original product of A if and only if A is abelian. If $A = \mathfrak{A}$ is a C^*-algebra, then the selfadjoint part $\mathfrak{A}_{\text{sa}} := \{a \in \mathfrak{A} \colon a = a^*\}$ is a *JC-algebra*, i.e. a real norm-closed subspace which is closed under the Jordan product $\bullet$. If $\mathfrak{A}_+ := \{a^* a \colon a \in \mathfrak{A}\}$ is the positive cone of $\mathfrak{A}$, then $\mathfrak{A}_+ = \{a^2 \colon a \in \mathfrak{A}_{\text{sa}}\}$. We have the following straightforward result.

Lemma I.2.1
Let $\mathfrak{A}, \mathfrak{B}$ be two C^*-algebras and $\phi \colon \mathfrak{A} \to \mathfrak{B}$ a selfadjoint map. The following are equivalent:

(a) ϕ is a Jordan homomorphism

(b) $\phi|_{\mathfrak{A}_{\mathrm{sa}}}\colon \mathfrak{A}_{\mathrm{sa}} \to \mathfrak{B}_{\mathrm{sa}}$ is a Jordan homomorphism

(c) $\phi(a^2) = \phi(a)^2$ for every $a \in \mathfrak{A}_{\mathrm{sa}}$

If one of the previous properties is satisfied, ϕ is a positive map.

Proof.
(a)⇔(b): the right implication is trivial. For the left one, let $a, a' \in \mathfrak{A}$. By decomposing them in their respective real and imaginary parts as usual, we get

$$\phi(a \bullet a') = \phi\big(\mathrm{Re}(a) \bullet \mathrm{Re}(a') - \mathrm{Im}(a) \bullet \mathrm{Im}(a') + i(\mathrm{Re}(a) \bullet \mathrm{Im}(a') + \mathrm{Im}(a) \bullet \mathrm{Re}(a'))\big)$$

$$\phi(a) \bullet \phi(a') = \phi\big(\mathrm{Re}(a) + i\mathrm{Im}(a)\big) \bullet \phi\big(\mathrm{Re}(a') + i\mathrm{Im}(a')\big)$$

Since $\mathrm{Re}(a), \mathrm{Re}(a'), \mathrm{Im}(a), \mathrm{Im}(a') \in \mathfrak{A}_{\mathrm{sa}}$ and $\phi|_{\mathfrak{A}_{\mathrm{sa}}}$ is a JC algebra homomorphism, $\phi(a \bullet a') = \phi(a) \bullet \phi(a')$.
(b)⇔(c): the right implication is trivial. For the left one, just notice that for $a, a' \in \mathfrak{A}_{\mathrm{sa}}$

$$2a \bullet a' = (a + a')^2 - a^2 - a'^2\,.$$

For the last statement, recall that $\mathfrak{A}_+ = \{a^2 \colon a \in \mathfrak{A}_{\mathrm{sa}}\}$, $\mathfrak{B}_+ = \{b^2 \colon a \in \mathfrak{B}_{\mathrm{sa}}\}$, then by (c) $\phi(\mathfrak{A}_+) \subseteq \mathfrak{B}_+$. □

Definition I.2.2
An invertible selfadjoint map $\phi\colon \mathfrak{A} \to \mathfrak{B}$ between two C^*-algebras is an *order isomorphism* if both ϕ and ϕ^{-1} are positive.

Remark I.2.3
If $\mathfrak{A} = \mathfrak{B} = \mathbb{C}^n$, it is not difficult to show that $\phi \in M_n(\mathbb{C})$ is an order isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$ if and only if ϕ is a non-negative *monomial* matrix, i.e. $\phi = DP$ for some diagonal matrix $D \in M_n(\mathbb{C})$ with strictly positive diagonal entries and $P \in \mathfrak{S}_n(\mathbb{C}) \subset GL_n(\mathbb{C})$ is a permutation matrix (that is why monomial matrices are often referred to as *generalized permutation matrices*).

Lemma I.2.4
Let $\mathfrak{A}, \mathfrak{B}$ be two C^*-algebras and $\phi\colon \mathfrak{A} \to \mathfrak{B}$ a selfadjoint map. The following are equivalent:

(a) ϕ is an order isomorphism

(b) ϕ is injective and $\phi(\mathfrak{A}_+) = \mathfrak{B}_+$

Proof.
Clearly, (a) implies (b). Viceversa, suppose that (b) holds. In particular, ϕ is positive. Now, let $b \in \mathfrak{B}$. Up to decomposing it in its real and imaginary parts, we can suppose it selfadjoint. Then, $b = b_+ - b_-$, with $b_\pm = \dfrac{|b| \pm b}{2} \in \mathfrak{B}_+$. Since $\phi(\mathfrak{A}_+) = \mathfrak{B}_+$, there exists a pair of elements $a, a' \in \mathfrak{A}_+$ s.t. $b = \phi(a - a') \in \phi(\mathfrak{A})$, that is ϕ is surjective. By hypothesis, it is also injective, hence it admits an inverse ϕ^{-1}, which satisfies $\phi^{-1}(\mathfrak{B}_+) = \mathfrak{A}_+$ i.e. is positive: ϕ is an order isomorphism. □

The deep connection between the two structures given to a C^*-algebra by the Jordan product $\bullet$ and the partial ordering $\leq$ induced by the positive cone is pointed out by Theorem 2.1.3 in [102] (p. 13-14).

Theorem I.2.5
Let $\mathfrak{A}, \mathfrak{B}$ be two C^*-algebras and $\phi\colon \mathfrak{A} \to \mathfrak{B}$ a selfadjoint map. If ϕ is a Jordan isomorphism, then it is an order isomorphism. Viceversa, if $\mathfrak{A}, \mathfrak{B}$ are also unital and ϕ is a *unital* order isomorphism, then it is a Jordan isomorphism.

Proof.
If ϕ is a Jordan isomorphism, then by Lemma I.2.1 both ϕ and ϕ^{-1} are positive i.e. ϕ is an order isomorphism. For the converse implication, unitality of $\mathfrak{A}, \mathfrak{B}, \phi$ is needed to exploit the Kadison-Schwarz inequality. Indeed, if ϕ is a unital order isomorphism, both ϕ and ϕ^{-1} are positive unital map, whence for each $a \in \mathfrak{A}_{\mathrm{sa}}$, $\phi(a)^2 \leq \phi(a^2)$ and

$$a^2 = (\phi^{-1}(\phi(a)))^2 \leq \phi^{-1}(\phi(a)^2) \leq \phi^{-1}(\phi(a^2)) = a^2 \,.$$

Therefore, $\phi(a^2) = \phi(a)^2$: ϕ is a Jordan isomorphism. □

Remark I.2.6
For instance, take $\mathfrak{A} = \mathfrak{B} = \mathcal{C}(\mathbb{T})$ and $\vartheta(f) := f(e^{i\frac{2\pi}{3}}\cdot)$. Then, $\vartheta \in \mathsf{Aut}(\mathcal{C}(\mathbb{T}))$ and $\vartheta^3 = I$. The unital linear map $\varphi := \dfrac{I+\vartheta}{2}$ is positive and invertible, with inverse $\varphi^{-1} = I - \vartheta + \vartheta^2$ which is not contractive (i.e. positive) since $\|\varphi^{-1}(z)\|_\infty = |1 - e^{i\frac{2\pi}{3}} + e^{i\frac{4\pi}{3}}| = 2 > \|z\|_\infty$. In retrospect, it all adds up: $f(z) := \mathrm{Re}(z) \in \mathcal{C}(\mathbb{T})_{\mathrm{sa}}$, but

$$\varphi(f^2) = \frac{\mathrm{Re}(e^{i\frac{5\pi}{3}}z^2)+2}{4}$$

$$\varphi(f)^2 = \frac{\mathrm{Re}(e^{i\frac{2\pi}{3}}z^2)+1}{8}$$

Hence, $\varphi(f^2) \geq \varphi(f)^2$ but $\varphi(f^2) \neq \varphi(f)^2$.

We can give a nice characterization of unital order automorphisms ($\mathfrak{A} = \mathfrak{B}$) in the abelian case. For the following result (appeared firstly in a special case on an unpublished note by Ionescu-Tulcea A. and C., then in a general setting, reported here, in [66]), given two C^*-algebras $\mathfrak{A}$, $\mathfrak{B}$ consider the convex set

$$\mathcal{B}(\mathfrak{A},\mathfrak{B})_{+,1} := \{\phi \in \mathcal{B}(\mathfrak{A},\mathfrak{B}) \,|\, \phi(\mathfrak{A}_+) \subseteq \mathfrak{B}_+, \phi(\mathbb{1}_{\mathfrak{A}}) = \mathbb{1}_{\mathfrak{B}}\} \,.$$

consisting of the unital positive (briefly, p.u.) maps from $\mathfrak{A}$ to $\mathfrak{B}$, with *extremal* points

$$\mathcal{E}\big(\mathcal{B}(\mathfrak{A},\mathfrak{B})_{+,1}\big) := \{\phi \in \mathcal{B}(\mathfrak{A},\mathfrak{B})_{+,1} \colon \phi = \lambda\phi_1 + (1-\lambda)\phi_2, \phi_i \in \mathcal{B}(\mathfrak{A},\mathfrak{B})_{+,1}, \lambda \in (0,1) \Rightarrow \phi = \phi_1 = \phi_2\} \,.$$

If $\mathfrak{A}$ is abelian, let $\Omega_{\mathfrak{A}}$ be its spectrum (character/maximal ideal space), a locally compact, T_2 space if endowed with the topology of poinwise convergence, which is compact iff $\mathfrak{A}$ is unital.

Theorem I.2.7
Let $\mathfrak{A}, \mathfrak{B}$ be unital abelian C^*-algebras and $\phi \colon \mathfrak{A} \to \mathfrak{B}$ a unital selfadjoint map. Then, the following are equivalent:

(a) $\phi \in \mathcal{E}\big(\mathcal{B}(\mathfrak{A},\mathfrak{B})_{+,1}\big)$

(b) $\phi \in \mathsf{Hom}(\mathfrak{A},\mathfrak{B})$

(c) $\phi = f^{\mathrm{t}}$ for some $f \in \mathcal{C}(\Omega_{\mathfrak{B}}, \Omega_{\mathfrak{A}})$, after identifying $\mathfrak{A}$, $\mathfrak{B}$ with $\mathcal{C}(\Omega_{\mathfrak{A}})$, $\mathcal{C}(\Omega_{\mathfrak{B}})$, respectively.

If (c) holds, f is uniquely determined, proper and closed. If $\Omega_{\mathfrak{A}}$ is also a *Stone space* (i.e. compact, T_2 and totally disconnected), the above properties are also equivalent to the following:

(d) $\phi \geq 0$ and preserves continuous indicator functions.

Proof.
See Theorem 2.1 in [66] (p. 270). □

Remark I.2.8
Examples of *Stone spaces* are (arbitrary products of) finite spaces with the discrete topology, Cantor spaces, profinite topological groups and Stone–Čech compactifications of any discrete space. As noticed by Phelps in [66] (p. 271), if $\Omega_{\mathfrak{A}}$ is not a Stone space, (d) never implies (a): $\Omega_{\mathfrak{A}}$ would admit a connected component C containing at least two points x, y, and $\phi := \dfrac{\mathrm{ev}_x + \mathrm{ev}_y}{2} \in \mathcal{S}(\mathcal{C}(\Omega_{\mathfrak{A}}))$ would preserve indicator functions though clearly not being extremal in $\mathcal{S}(\mathcal{C}(\Omega_{\mathfrak{A}})) \cong \mathcal{M}_1(\Omega_{\mathfrak{A}})$ (state space of $\mathcal{C}(\Omega_{\mathfrak{A}})$, or equivalently, probability measure space on $\Omega_{\mathfrak{A}}$).

Recalling once more that in the abelian case the Jordan and the usual products coincide, we can infer the following corollary, by putting together Theorem I.2.5 and Theorem I.2.7.

Corollary I.2.9
Let $\mathfrak{A}$ be a unital abelian C^*-algebras and $\phi \in \mathcal{B}(\mathfrak{A})_{+,1}$ an invertible p.u. map. The following are equivalent:

(a) $\phi \in \mathsf{Aut}(\mathfrak{A})$

(b) ϕ is an order automorphism

(c) $\phi \in \mathcal{E}\big(\mathcal{B}(\mathfrak{A})_{+,1}\big)$

(d) there exists $f \in \mathsf{Homeo}(\Omega_{\mathfrak{A}})$ s.t. $\phi = f^{\mathrm{t}}$

If (d) holds, f is uniquely determined. If $\Omega_{\mathfrak{A}}$ is totally disconnected, the next assertion is also equivalent to the above:

(e) ϕ permutes the continuous indicator functions on $\Omega_{\mathfrak{A}}$

I.2.3 C^*-dynamical systems

In the whole chapter, for (*discrete*) C^**-dynamical system* (or simply C^*-system), we shall mean a triple $(\mathfrak{A}, \phi, M)$, where $\mathfrak{A}$ is a C^*-algebra, ϕ is a c.p.u. map acting on $\mathfrak{A}$ via its powers, and M is the monoid $\mathbb{N}$ or $\mathbb{Z}$. By definition, the case relative to the group $\mathbb{Z}$ corresponds to ϕ being a *-automorphism. In this context, we talk about (microscopically) reversible C^*-systems.[2] Alternative terms are "conservative", "Hamiltonian" and "unitary" systems. We will treat instead the dissipative cases, when $M = \mathbb{N}$ and Φ is in general not invertible. The simplified notation $(\mathfrak{A}, \phi)$ will stand for the triple $(\mathfrak{A}, \phi, \mathbb{N})$.
Given a Banach space $\mathfrak{X}$, let $T \in \mathcal{B}(\mathfrak{X})$ such that $r(T) = 1$. It is customary to set the space of the *almost periodic* elements of T as

$$\mathrm{AP}(T) := \overline{\mathrm{span}_{\mathbb{C}}\{x \in \mathfrak{X} \colon Tx = \lambda x \text{ for some } \lambda \in \sigma_{\mathrm{per}}(T)\}}$$

(See *e.g.* [32] for a standard situation.)
We point out the following

Remark I.2.10
Let $\phi : \mathfrak{A} \to \mathfrak{A}$ be an irreducible c.p.u. map. By Proposition 3.2 in [42], $\mathrm{AP}(\phi) \subset P_\phi(\mathfrak{A})$ is a C^*-subalgebra of $\mathfrak{A}$, and the restriction $\phi|_{\mathrm{AP}(\mathfrak{A})}$ of ϕ to $\mathrm{AP}(\mathfrak{A})$ is automatically a *-automorphism.

[2] Physical systems that are macroscopically irreversible, but microscopically reversible, typically describe temperature states (or, equivalently, states satisfying the Kubo-Martin-Schwinger boundary condition) since their dynamics is generated by unitary operators (see *e.g.* [87]).

Here, we are using the notion of irreducibility in [42] (Definition 2.2, p. 312): a positive operator $\phi\colon \mathfrak{A} \to \mathfrak{A}$ is *irreducible* if $\mathfrak{A}$ does not admit any non-trivial ϕ-invariant C^*-subalgebras A s.t. $A \cap \mathfrak{A}_+$ is a face of $\mathfrak{A}$ (equivalently, there in no non-trivial closed ϕ-invariant face of $\mathfrak{A}_+$).[3] In the upcoming subsection, we shall see that this definition of irreducibility is equivalent to the one normally used for positive matrices.

I.2.4 Stochastic matrices

When $\mathfrak{A} = \mathbb{C}^n$, the u.(c.)p. self-maps are exactly the *stochastic* $n \times n$ *matrices*, i.e. the non-negative square matrices of order n with row sums 1. In formula,

$$\mathcal{B}(\mathfrak{A})_{+,1} = \left\{ M \in M_n(\mathbb{C})\colon m_{ij} \geq 0, \sum_{k=1}^{n} m_{ik} = 1 \, (i,j = 1, \dots, n) \right\} .$$

Given a stochastic matrix S, the C^*-system $(\mathbb{C}^n, S)$ models the dynamics of a *Markov chain* (for a reference, see *e.g.* [100]). The structure of stochastic matrices is briefly outlined in Section I.4. Here, we recall their basic properties. Unitality of S tells us that $1 := \mathbb{1}_{\mathbb{C}^n} = \begin{bmatrix} 1 & \dots & 1 \end{bmatrix}^t$ is a right eigenvector of S pertaining to the eigenvalue $1 \in \sigma_{\text{per}}(S)$. Roughly speaking, it means that at each step of the transition in the Markov chain (i.e. after the repeated application of S on vectors of $\mathbb{C}^n$) probability must be conserved. On the other hand, any non-negative row-vector $\begin{bmatrix} \pi_1 & \pi_2 & \cdot & \cdot & \cdot & \pi_n \end{bmatrix}$ with $\displaystyle\sum_{i=1}^{n} \pi_i = 1$, which is also a left eigenvector corresponding to the eigenvalue 1, is a *stationary distribution* for the Markov chain. Algebraic and geometric multiplicities of the left and right eigenvalue 1 always coincide.

A square non-negative matrix A of order n is said to be *irreducible* if there exists no $P \in S_n(\mathbb{C})$ such that $\operatorname{ad}_P(A) = PAP^{-1} = \begin{bmatrix} A_1 & B \\ O & A_2 \end{bmatrix}$, where A_i is a square matrix of order $1 \leq n_i < n$ (see *e.g.* [71], [100]). For the convenience of the reader, we show that the definition of irreducibility provided in Definition 2.2 of [42] and reported in the previous subsection coincides with the one given here for non-negative matrices.

Proposition I.2.11
Let $A \in M_n(\mathbb{C})$ be non-negative. Then, it is irreducible if and only if the only A-invariant faces of the positive cone $\mathbb{C}^n_+ = \displaystyle\bigoplus_{j=1}^{n} \mathbb{R}^+ e_j$ are $\{0\}$ and the whole $\mathbb{C}^n_+$.

Proof.
Suppose A irreducible, i.e. there exists a face $\mathcal{F} := \displaystyle\bigoplus_{j \in J} \mathbb{R}^+ e_j$ for some set $J \neq \varnothing, \{1, \dots, n\}$ which ios invariant under A, that is $A(\mathcal{F}) \subset \mathcal{F}$. Let $\sigma \in \mathfrak{S}_n$ be any permutation of $\{1, \dots, n\}$ s.t. $\sigma(J) = \{1, \dots, |J|\}$. By setting $P_\sigma(e_k) := e_{\sigma(k)}$ for $k = 1, \dots, n$, $P_\sigma \in S_n(\mathbb{C})$ is s.t. $P_\sigma A P_\sigma^{-1} = \begin{bmatrix} A_1 & B \\ O & A_2 \end{bmatrix}$ with A_1 a square matrix of size $1 \leq n_1 = |J| \leq n-1$ and A_2 a square matrix of size $1 \leq n_2 \leq n-1$: a contradiction. Viceversa, suppose that there exists a permutation matrix $P \in S_n(\mathbb{C})$ such that $PAP^{-1} = \begin{bmatrix} A_1 & B \\ O & A_2 \end{bmatrix}$ as above. Consider the set $\{e_j\}_{j \in J} := \{P^{-1} e_i\}_{i=1}^{n_1}$. Then, $\displaystyle\bigoplus_{j \in J} \mathbb{R}^+ e_j$ is a non-trivial A-invariant face of $\mathbb{C}^n_+$, which is again a contradiction. □

[3] Recall that a *face* $\mathcal{F}$ of $\mathfrak{A}_+$ is a subcone of $\mathfrak{A}_+$ such that if $a \in \mathfrak{A}_+$, $b \in \mathcal{F}$ satisfy $a \leq b$, then $a \in \mathcal{F}$.

I.3. C.p.u. maps on finite-dimensional C^*-algebras

The present section is devoted to basic results on which is based the forthcoming analysis for stochastic matrices. We report those for the convenience of the reader.

Proposition I.3.1
Let $\mathfrak{A}$ be a finite-dimensional, unital C^*-algebra and ϕ a c.p.u. self-map on $\mathfrak{A}$. Then, there exists a subsequence $\{n_j\}_{j\in\mathbb{N}} \subset \mathbb{N}$ s.t. $\lim\limits_{j\to+\infty} \phi^{n_j} = P_\phi$ in norm. In particular, P_ϕ is a c.p.u. projection and the injective operator system $P_\phi(\mathfrak{A})$ is a (unital, finite-dimensional) C^*-algebra when endowed with the Choi-Effros product

$$a \circ b := P_\phi(ab), \quad a, b \in P_\phi(\mathfrak{A})\,. \tag{I.1}$$

Proof.
The Jordan-Chevalley decomposition of ϕ gives $\phi = \sum\limits_{\lambda\in\sigma_{\text{per}}(\phi)} \lambda E_\lambda + Q_\phi\phi$, with $E_\lambda E_\mu = \delta_{\lambda,\mu}E_\lambda$, $\sum\limits_{\lambda\in\sigma_{\text{per}}(\phi)} E_\lambda = P_\phi$. Since $\sigma_{\text{per}}(\phi)$ is a finite subset (not necessarily a subgroup) of $\mathbb{T}$, there must exist a subsequence $\{n_j\}_{j\in\mathbb{N}}$ of natural numbers such that $\lim\limits_{j\to+\infty} \lambda^{n_j} = 1$ for every $\lambda \in \sigma_{\text{per}}(\phi)$. Then, by Proposition 3.1 in [34] (p. 110),

$$\lim_{j\to+\infty} \phi^{n_j} = \lim_{j\to+\infty} \Big(\sum_{\lambda\in\sigma_{\text{per}}(\phi)} \lambda E_\lambda + Q_\phi\phi \Big)^{n_j} = \lim_{j\to+\infty} \Big(\sum_{\lambda\in\sigma_{\text{per}}(\phi)} \lambda E_\lambda \Big)^{n_j} + \lim_j \big(Q_\phi \phi^{n_j}\big) =$$

$$= \lim_{j\to+\infty} \sum_{\lambda\in\sigma_{\text{per}}(\phi)} \lambda^{n_j} E_\lambda = \sum_{\lambda\in\sigma_{\text{per}}(\phi)} \Big(\lim_{j\to+\infty} \lambda^{n_j}\Big) E_\lambda = \sum_{\lambda\in\sigma_{\text{per}}(\phi)} E_\lambda = P_\phi\,.$$

In particular, P_ϕ is a c.p.u. projection. Thanks to Theorem 3.1 in [16], Equation I.1 defines a C^*-product on $P_\phi(\mathfrak{A})$. □

Remark I.3.2
Under the assumptions of Proposition I.3.1, evidently

$$P_\phi(\mathfrak{A}) = \text{span}_{\mathbb{C}}\{a \in \mathfrak{A} \colon \phi(a) = \lambda(a) \text{ for some } \lambda \in \sigma_{\text{per}}(\phi)\}\,.$$

In the hypothesis of the previous proposition, we set $\mathfrak{A}_\phi := (P_\phi(\mathfrak{A}), \mathbb{1}_{\mathfrak{A}}, *, \circ, \|\cdot\|_{\mathfrak{A}})$ the C^*-algebra induced by the c.p.u. self-map ϕ using the Choi-Effros construction.

For completeness, we conclude this section with a well-known fact: the mean ergodicity of c.p.u. maps on a finite-dimensional C^*-algebra.

Proposition I.3.3
Let $\phi : \mathfrak{A} \to \mathfrak{A}$ be a c.p.u. map on the finite-dimensional C^*-algebra $\mathfrak{A}$. Then,

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi^k = E_1\,,$$

the projection onto the fixed point subspace $\mathfrak{A}^\phi$. Moreover, E_1 is a c.p.u. map.

Proof.
By performing the same calculations in the above proof, we get

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi^k = E_1 + \sum_{\lambda\in\sigma_{\text{per}}(\phi)\setminus\{1\}} \frac{1}{1-\lambda} \lim_{n\to+\infty} \frac{1-\lambda^n}{n} E_\lambda = E_1$$

since $|1-\lambda^n| \leq 2$. For the proof of the second part of the statement see [33], Theorem 2.1 (p. 182). □

I.4. Stochastic matrices and persistent C^*-systems

Let S be a stochastic matrix as defined in Subsection I.2.4. Up to row-column permutations, any stochastic matrix $S \in M_n(\mathbb{C})$ has the following canonical form

$$S = \begin{bmatrix} B_{00} & B_{01} & \cdot & \cdot & \cdot & B_{0t} \\ 0 & B_{11} & 0 & \cdot & \cdot & 0 \\ \cdot & 0 & B_{22} & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & B_{tt} \end{bmatrix}. \tag{I.2}$$

(See [100], Proposition 8.8 and the discussion after Proposition 9.2).
Here, B_{00} is a square strictly sub-stochastic matrix associated to the transient indices (which is the empty matrix if and only if the subset of such transient indices is empty[4]), while the square-block matrices B_{kk}, $k = 1, \ldots, t$ are irreducible. Each of them is the transition matrix of an ergodic component of the Markov chain generated by S. (I.2) is said to be the *reduced form* of S and, if there is no transient indices, S is said to be *completely reducible*. The following theorem collects some crucial properties of (I.2).

Theorem I.4.1
Referring to the canonical form (I.2) of a stochastic matrix $S \in M_n(\mathbb{C})$,

(i) for each $k = 1, \ldots, t$, $\sigma_{\text{per}}(B_{kk}) = \{\omega \in \mathbb{C} \colon \omega^{d_k} = 1\} \cong \mathbb{Z}_{d_k} \leq \mathbb{T}$ with d_k index of imprimitivity of B_{kk}, and the multiplicity of all peripheral eigenvalues is always 1;

(ii) $\sigma(S) = \bigcup_{k=0}^{t} \sigma(B_{kk})$;

(iii) $\sigma_{\text{per}}(S) = \bigcup_{k=1}^{t} \sigma_{\text{per}}(B_{kk})$.

Proof.
(i) follows by Theorem I.6.5 in [100], where $\sigma_{\text{per}}(B_{kk})$ coincides with the d_k-th roots of the unity, d_j being the index of imprimitivity of B_{kk}, see [100], Section I.9.[5]
(ii) is well known, see *e.g.* [107], Section 2.3.[6]
(iii) follows from (ii) because $\sigma_{\text{per}}(B_{00}) = \varnothing$. Indeed, if B_{00} had an eigenvalue λ with $|\lambda| = 1$, Proposition I.9.3 in [100] would not hold. □

Given a stochastic matrix $S \in M_n(\mathbb{R})$, the associated Markov chain is nothing else than a (commutative, finite-dimensional) C^*-system $(\mathbb{C}^n, S)$, where the matrix S generates, via its non-negative powers, the action of the monoid $\mathbb{N}$. On the other hand, by appling Proposition I.3.1 to $\mathfrak{A} := \mathbb{C}^n$ and $\phi := S$, the linear space $\mathfrak{A}_S := P_S(\mathbb{C}^n)$ is in fact a (unital, abelian) C^*-algebra with the new Choi-Effros product $\circ$. We are thus ready to show our main result: $(\mathfrak{A}_S, S|_{\mathfrak{A}_S})$ provides a genuine *conservative* C^*-system, and thus the action of $S|_{\mathfrak{A}_S}$ can now be extended to negative powers.

[4] In the language of Markov chains, the transient indices are associated to the so-called transient (or inessential) "states".

[5] If the index of imprimitivity d_k of a block B_{kk} in the reduced form (I.2) is 1, then B_{kk} is said to be *primitive*.

[6] As witnessed by the matrix $\begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 0 & 2/3 & 1/3 \\ 0 & 0 & 1 \end{bmatrix}$, $B_{00} = \begin{bmatrix} 1/2 & 1/4 \\ 0 & 2/3 \end{bmatrix} = \begin{bmatrix} B_{00}^{00} & B_{00}^{01} \\ 0 & B_{00}^{11} \end{bmatrix}$ can be further reduced. Yet, $\sigma(B_{00}) = \{1/3, 2/3\} = \sigma(B_{00}^{00}) \cup \sigma(B_{00}^{11})$.

Theorem I.4.2
With the above notation, $S|_{\mathfrak{A}_S}$ is an order automorphism, and hence $(\mathfrak{A}_S, S|_{\mathfrak{A}_S}, \mathbb{Z})$ is a conservative C^*-system.

Proof.
By Jordan-Chevalley decomposition, $S|_{\mathfrak{A}_S} = \sum_{\lambda \in \sigma_{\text{per}}(S)} \lambda E_\lambda$. Since by Theorem I.4.1,

$$\sigma_{\text{per}}(S) = \bigcup_{k=1}^{t} \sigma_{\text{per}}(B_{kk}) = \bigcup_{k=1}^{t} \left\langle e^{i\frac{2\pi}{d_k}} \right\rangle$$

$\left(S|_{\mathfrak{A}_S}\right)^{\text{lcm}(d_1,\ldots,d_t)} = \text{id}_{\mathfrak{A}_S}$. Therefore, $\left(S|_{\mathfrak{A}_S}\right)^{-1} = \left(S|_{\mathfrak{A}_S}\right)^{\text{lcm}(d_1,\ldots,d_t)-1}$ which is manifestly positive.[7] By Corollary I.2.9, $S|_{\mathfrak{A}_S} \in \mathsf{Aut}(\mathfrak{A}_S)$ and hence $(\mathfrak{A}_S, S|_{\mathfrak{A}_S}, \mathbb{Z})$ is a conservative C^*-system. □

We end the section with some considerations. By taking into account Proposition I.2.11, point 2 in Proposition 3.2 of [42], and lastly (iii) in Theorem I.4.1, we conclude that in the irreducible cases, hence in all completely reducible ones, the Choi-Effros product Equation I.1 coincides with the original one. On the other hand, we know that there are examples, necessarily admitting transient indices, for which the original product must be changed, see *e.g.* [34], Section 6. Therefore, one might conclude that the cases for which the original product should be changed is connected with the presence of transient indices. Unfortunately, also this conjecture does not hold in general. For instance, when all the imprimitivity indices d_j, $j = 1, 2, \ldots, n$, of the square-block matrices in (I.2) are 1, $\sigma_{\text{per}}(S) = \{1\}$ with multiplicity n and the original product need not to be changed. All things considered, the cases for which the original product might be replaced with the Choi-Effros one have to be found among those with a non-empty set of transient indices and at least one ergodic imprimitive component $B_{j_o j_o}$.

I.5. Generalizations of the main result

A year after the publication of our work, in September 2023 Bhat, Kar and Talwar prove the following result.

Theorem I.5.1 (Bhat, Kar, Talwar)
Let $\mathfrak{A}$ be a finite-dimensional, unital C^*-algebra. If $\phi\colon \mathfrak{A} \to \mathfrak{A}$ is a c.p.u. self-map such that $\sigma(\phi) \subseteq \mathbb{T}$, then $\phi \in \mathsf{Aut}(\mathfrak{A})$.

Proof.
Without loss of generality, we can take $\mathfrak{A} = \bigoplus_{n=1}^{N} M_{d_n}(\mathbb{C})$ for some $d_1, \ldots, d_N, N \geq 1$. Let $\mathcal{H} := \bigoplus_{n=1}^{N} \mathbb{C}^{d_n}$ and $(P_n\colon \mathcal{H} \twoheadrightarrow \mathbb{C}^{d_n})_{n=1}^{N}$ the corresponding family of pairwise orthogonal projections. Then, $\mathcal{B}(\mathcal{H}) = M_d(\mathbb{C})$ where $d := \sum_{n=1}^{N} d_n$. Let

$$\begin{aligned} \widetilde{\phi}\colon \mathcal{B}(\mathcal{H}) &\to \mathcal{B}(\mathcal{H}) \\ X &\mapsto \sum_{n=1}^{N} \phi(P_n X P_n) \end{aligned}$$

[7] If all blocks B_{jj}, $j = 1, \ldots, n$, are primitive, that is $d_j = 1$, then on one hand $S|_{\mathfrak{A}_S} = I_{\mathfrak{A}_S}$, while on the other hand $\text{l.c.m.}(d_1, \ldots, d_n) - 1 = 0$ which means $\left(S|_{\mathfrak{A}_S}\right)^{-1} = I_{\mathfrak{A}_S} = \left(S|_{\mathfrak{A}_S}\right)^{\text{l.c.m.}(d_1,\ldots,d_n)-1}$.

Then, $\widetilde{\phi}$ is a c.p.u. self-map on $\mathcal{B}(\mathcal{H})$ s.t. $P_{\widetilde{\phi}}(\mathcal{B}(\mathcal{H})) \cong P_\phi(\mathfrak{A}) = \mathfrak{A}$ (this last equality is due to $\sigma(\phi) \subseteq \mathbb{T}$ whence $\sigma(\phi) = \sigma_{\mathrm{per}}(\phi)$). By Theorem 2.10 in [8] (p. 199), $\widetilde{\phi}|_{P_{\widetilde{\phi}}(\mathcal{B}(\mathcal{H}))} \in \mathsf{Aut}(P_{\widetilde{\phi}}(\mathcal{B}(\mathcal{H})))$, and hence $\phi \in \mathsf{Aut}(\mathfrak{A})$. □

Corollary I.5.2
Let $\mathfrak{A}$ be a finite-dimensional, unital C^*-algebra. If $\phi\colon \mathfrak{A} \to \mathfrak{A}$ is a c.p.u. self-map, then $\phi|_{\mathfrak{A}_\phi} \in \mathsf{Aut}(\mathfrak{A}_\phi)$ and hence $(\mathfrak{A}_\phi, \phi|_{\mathfrak{A}_\phi}, \mathbb{Z})$ is a conservative C^*-system.

Proof.
Since $\sigma(\phi|_{\mathfrak{A}_\phi}) = \sigma_{\mathrm{per}}(\phi) \subseteq \mathbb{T}$, the assertion follows from Theorem I.5.1. □

Remark I.5.3
The hypothesis of *complete* positivity in Theorem I.5.1 cannot be weakened to mere positivity. The transpose map $T\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ $(n \geq 2)$ is an involutive, unital, positive map which is not 2-positive (on the contrary, it is 2-*copositive*). Its spectrum is $\sigma(T) = \{\pm 1\} \subset \mathbb{T}$, with spectral subspaces $M_n(\mathbb{C})_1 = \{A \in M_n(\mathbb{C})\colon A \text{ symmetric}\}$ of dimension $\frac{n}{2}(n+1)$ and $M_n(\mathbb{C})_{-1} = \{A \in M_n(\mathbb{C})\colon A \text{ anti-symmetric}\}$ of dimension $\frac{n}{2}(n-1)$. However, it is merely a *-anti-automorphism of $M_n(\mathbb{C})$ (i.e. a *-isomorphism from $M_n(\mathbb{C})$ to its opposite algebra $M_n(\mathbb{C})^{\mathrm{op}}$).

One might raise the question whether Theorem I.5.1 is just a special case of a more general fact, firstly conjectured in [34] (Conjecture 5.5, p. 120):

Let ϕ be a c.p.u. self-map on a unital C^*-algebra $\mathfrak{A}$ s.t. $\sigma(\phi) \subset \mathbb{T}$. Then, $\phi \in \mathsf{Aut}(\mathfrak{A})$.

With the enlightening suggestions of Glück J. (at the end of January 2024), we have been able to produce an ∞-dimensional example for which Theorem I.5.1 abruptly fails to hold. Consider the non-separable, abelian W^*-algebra $\ell^\infty(\mathbb{Z}) \cong \mathcal{C}_b(\mathbb{Z}) \cong M(\mathcal{C}_0(\mathbb{Z})) \cong \mathcal{C}(\beta\mathbb{Z})$, where $\beta\mathbb{Z}$ is the Stone-Čech compactification of $\mathbb{Z}$ (an example of Stone space, indeed hyperstonean, see Remark I.2.8). Notice that the well-known *bilateral shift*, defined as $(Sx)_n := x_{n-1}$ $(n \in \mathbb{Z}$, $x \in \ell^\infty(\mathbb{Z}))$ is a *-automorphism of $\ell^\infty(\mathbb{Z})$ and $\sigma(S) = \sigma_{\mathrm{p}}(S) = \mathbb{T}$, with spectral subspaces of the form $\ell^\infty(\mathbb{Z})_z = \mathrm{span}_{\mathbb{C}}\{n \mapsto z^{-n}\}$ for every $z \in \mathbb{T}$ ($\sigma_{\mathrm{p}}(S)$ denotes the point spectrum of S, namely the set of its eigenvalues). A modified version of S does the job we want: we will see that it does not belong to $\mathsf{Aut}(\ell^\infty(\mathbb{Z}))$, though having exactly the same spectrum as S. Let

$$\begin{aligned}\phi\colon \ell^\infty(\mathbb{Z}) &\to \ell^\infty(\mathbb{Z})\\ x &\mapsto \phi(x)\colon \begin{cases} 1 \mapsto \dfrac{x_0 + x_{-1}}{2} \\ n \mapsto x_{n-1} \quad (n \neq 1)\end{cases}\end{aligned}$$

with k^{th}-power $(k \geq 1)$

$$\begin{aligned}\phi^k\colon \ell^\infty(\mathbb{Z}) &\to \ell^\infty(\mathbb{Z})\\ x &\mapsto \phi^k(x)\colon \begin{cases} n \mapsto \dfrac{x_{n-k} + x_{n-k-1}}{2} & (1 \leq n \leq k) \\ n \mapsto x_{n-k} & \text{otherwise}\end{cases}\end{aligned}$$

Then, ϕ is an invertible, normal (i.e. ultraweakly continuous) c.p.u. self-map of $\ell^\infty(\mathbb{Z})$ (ϕ is an example of *Markov operator* on $\mathcal{C}(\beta\mathbb{Z})$, as defined and thoroughly examined in [40]). Its pre-adjoint is

$$\phi_* \colon \ell^1(\mathbb{Z}) \to \ell^1(\mathbb{Z})$$
$$x \mapsto \phi_*(x)\colon \begin{cases} -1 \mapsto x_0 + \dfrac{x_1}{2} \\ 0 \mapsto \dfrac{3x_1}{2} \\ n \mapsto x_{n+1} \quad (n \neq -1, 0) \end{cases}$$

and the k^{th} power of its inverse ($k \geq 1$) is

$$\phi^{-k} \colon \ell^\infty(\mathbb{Z}) \to \ell^\infty(\mathbb{Z})$$
$$y \mapsto \phi^{-k}(y)\colon \begin{cases} n \mapsto (-1)^{n+k}\left(y_0 + 2\sum_{j=1}^{n+k}(-1)^j y_j\right) \quad (1-k \leq n \leq 0) \\ n \mapsto y_{n+k} \quad \text{otherwise} \end{cases}$$

By Gel'fand-Beurling spectral radius formula,

$$1 \leq r(\phi^{-1}) = \lim_{k\to+\infty} \|\phi^{-k}\|^{1/k} \leq \lim_{k\to+\infty} (2k+1)^{1/k} = 1$$

whence $\sigma(\phi) \subseteq \mathbb{T}$. Even more, $\sigma(\phi) = \sigma_{\mathrm{p}}(\phi) = \mathbb{T}$. Indeed, for every $z \in \mathbb{T}$, $\ell^\infty(\mathbb{Z})_z = \operatorname{span}_{\mathbb{C}}\{x_z\}$ where

$$x_z : \begin{cases} n \mapsto z^{-n} \quad (n \leq 0) \\ n \mapsto \left(\dfrac{1+z}{2}\right) z^{-n} \quad (n \geq 1)\,. \end{cases}$$

However, clearly $\phi \notin \mathsf{Aut}(\ell^\infty(\mathbb{Z}))$: $\phi(\delta_0^2) = \phi(\delta_0) = \dfrac{\delta_1}{2} \neq \dfrac{\delta_1}{4} = \phi(\delta_0)^2$. By Corollary I.2.9, this is due to the fact that although ϕ is positive, ϕ^{-1} is not: $\phi^{-1}(\delta_1) = 2\delta_0$ whence $\|\phi^{-1}\| \geq 2$. Nonetheless, we notice that by Theorem 2.12 in [109], we have at least $\phi \in \mathsf{Aut}(\mathrm{AP}(\phi), \circ_{\mathrm{AP}})$ where

$$\mathrm{AP}(\phi) = \overline{\operatorname{span}_{\mathbb{C}}\{x \in \ell^\infty(\mathbb{Z})\colon \phi(x) = \lambda x \text{ for some } \lambda \in \mathbb{T}\}} = \overline{\operatorname{span}_{\mathbb{C}}\{\ell^\infty(\mathbb{Z})_z \colon z \in \mathbb{T}\}} \subsetneq \ell^\infty(\mathbb{Z})$$

(cf. Subsection I.2.3) is a C^*-algebra if endowed with the product

$$x_z \circ_{\mathrm{AP}} x_w := \underset{k\to+\infty}{s\text{-}\lim} \frac{\phi^k(x_z x_w)}{(zw)^k}, \quad z, w \in \mathbb{T}$$

and the C^*-norm and involution inherited from $\ell^\infty(\mathbb{Z})$. Observe that $x_z \circ_{\mathrm{AP}} x_w = x_{zw}$ for every $z, w \in \mathbb{T}$. Indeed, the strong operator convergence on $\ell^\infty(\mathbb{Z})$ is merely the pointwise one and for each $k \geq 1$

$$\frac{\phi^k(x_z x_w)}{(zw)^k} \colon \mathbb{Z} \to \mathbb{C}$$
$$n \mapsto (zw)^{-n} \quad (n \leq 0)$$
$$n \mapsto \left(\frac{1+zw}{2}\right)(zw)^{-n} \quad (1 \leq n \leq k)$$
$$n \mapsto \frac{(1+z)(1+w)}{4}(zw)^{-n} \quad (n \geq k+1)\,.$$

I.6. Physical remarks

In Classical Physics, time evolution is described by a suitable differential equation, and studying the possible superposition of a *persistent* and a *transient* part turns out to be very natural. In all these systems, such an analysis simply describes a partition of relevant physical quantities into the part which is vanishing (transient part) and the one surviving (persistent part) when $t \to +\infty$, provided such a splitting can be performed.[8] This means that, for the investigation of a long-time behaviour, only the persistent part is substantial and, clearly, only the properties of the surviving portion of the original dynamical system are encoded by the (surviving, i.e. restricted) time evolution. As a useful example when building electrical measurement tools, we mention the forced RLC circuit.[9]
To summarize, in the classical situation, such a simplified notion of decoherence consisting in the splitting into a transient and a persistent part is well understood. With the arrival of Quantum Mechanics, the precise axiomatization of the measurement procedure assumed a fundamental role, thus making things appear much more complicated. One of the axioms contemplates that the set of observable quantities is modelled by some suitable Jordan algebra. Since the structure of a Jordan algebra is far from being completely understood, to provide significant models and avoid many technical troubles, it is usually assumed that such a Jordan algebra is the selfadjoint part of a C^*-algebra, see *e.g.* [90], Section 2.
By coming back to the universally accepted (quantum) decoherence, on one hand it takes place when the whole system can be described by the superposition of the multiplicative domain, automatically a C^*-algebra under its own multiplicative operation, and the remaining part disappearing in time. On the other hand, as it happens for gapped c.p.u. maps, there are very simple examples which do not satisfy this standard definition of decoherence. However, the part pertaining to the peripheral spectrum can still be separated by the remainder: the former provides a dynamical (not necessarily C^*) system which survives, the latter is inessential in the long-time behaviour.
The aim of [36] was to show that Markov chains, a relevant class of commutative examples, encode a conservative C^*-dynamical system after isolating the persistent part from the transient one, and equipping the former with a new product. Such a conservative dynamical system is in general larger than that consisting merely by the multiplicative domain. This result appears as a relevant step to provide a partial answer to the general, and currently unsolved, decoherence problem.

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[8] Here, the parameter t describes the time evolution.

[9] Here, R, L and C stand for "resistance", "inductance" and "capacitance", respectively.

Chapter II

Graded C^*-algebras and twisted tensor products: a state space approach

II.1. Introduction

The tensor product $X \odot Y$ of two (possibly, topological) linear spaces X and Y is one of the most recurring constructions in linear algebra and functional analysis. It shows really interesting features, as well as many technical problems often difficult to deal with. If X and Y have some additional structure, new properties can naturally be investigated. For instance, if the involved spaces are Hilbert ones, say $\mathcal{H}$ and $\mathcal{K}$, the Hilbert tensor product, usually denoted by $\mathcal{H} \otimes \mathcal{K}$, has countless applications in Fourier analysis, ergodic theories and quantum statistical mechanics. When $X \equiv \mathfrak{A}$ and $Y \equiv \mathfrak{B}$ are C^*-algebras, $\mathfrak{A} \odot \mathfrak{B}$ is naturally endowed with an algebraic structure making it an involutive algebra, denoted also by $\mathfrak{A} \otimes \mathfrak{B}$. Since $\mathfrak{A}$ and $\mathfrak{B}$ now have a norm topology, it is natural to study the possible C^*-norms which can defined on $\mathfrak{A} \otimes \mathfrak{B}$, thus exhibiting its C^*-completions. Among such norms, the minimal and the maximal ones are, respectively, the smallest and the greatest in the family of all C^*-norms on $\mathfrak{A} \otimes \mathfrak{B}$. Plus, the von Neumann tensor product between two W^*-algebras $\mathfrak{M}$ and $\mathfrak{N}$ is easily built from faithful representations $\pi^{(\mathfrak{M})}$ and $\pi^{(\mathfrak{N})}$ as $\mathfrak{M}\overline{\otimes}\mathfrak{N} := \left(\pi^{(\mathfrak{M})}(\mathfrak{M}) \otimes \pi^{(\mathfrak{N})}(\mathfrak{N})\right)''$, and it is seen that such a construction does not depend on the chosen pair $\pi^{(\mathfrak{M})}$, $\pi^{(\mathfrak{N})}$. The investigation of the possible uniqueness of the C^*-norm on tensor products provides a possible definition of *nuclearity*, which is also connected to *injectivity*, both very important notions naturally arising in the theory of operator algebras. We mention the nice paper [10] devoted to the systematic study of possible C^*-norms on a tensor product.

Tensor products are also deeply connected to the notion of *independence* in Quantum Probability and Quantum Field Theory. Even in Classical Probability, the usual notion of independence is indeed that of *tensor independence*. In the quantum framework, various notions of independence are analysed in [38] under some natural conditions: freeness, (anti-)monotonicity and booleanness. By relaxing the conditions in [38], much more natural notions of independence (such as the $\mathbb{Z}_n$-graded independence) are given in [41], where $n = 1$ collapses to the tensor one. In Quantum Field Theory, independence is connected to *Einstein's causality*. Einstein's causality simply asserts that it is possible to exchange signals only between points connected by a time-like worldline. Such a fundamental principle is encoded in a set of reasonable axioms to describe the majority of models as exposed in the seminal paper [44]. As a consequence, algebras of observables localized in causally separated regions are independent and partial states associated to such regions are essentially uncoupled, see *e.g.* [69]. This leads to the celebrated *split property* (*e.g.* [13]) which is asserted to be satisfied for most of relevant models. Nonetheless, it is well known that a first classification of elementary particles is provided in terms of their spin. Such a classification is fundamental in quantum field theory and statistical mechanics. Particles having half-integer spin, named *fermions*, obey the Fermi-Dirac statistics, whereas those with integer spin, called *bosons*, obey the Bose-Einstein one. This is the fundamental spin-statistics theorem, pervading all quantum physics. Fermi particles are satisfactorily described by Fermi fields which enjoy the so-called Canonical Anticommutation Relations (CAR's) when they are causally separated, see *e.g.* [103], and [87] for the applications to statistical mechanics. Causally separated algebras of such Fermi fields provide a kind of suitably "twisted" (w.r.t. to product and involution) tensor product, directly represented on a Hilbert space. In this scheme, charged Fermi fields localized in spatially separated regions anticommute with each other.

As for the Canonical Commutation Relations (CCR's) enjoyed by Bose fields, abstractly represented by the usual C^*-tensor product, one could search for the systematic construction of a general C^*-model for CAR's, which might be called *Fermi C^*-tensor product*. Quite evidently, such a twisted construction implies many more complications, even at a purely algebraic level, compared to the usual one. Notice that the interest in Fermi models is increasing also due to their applications in statistical mechanics and, consequently, in quantum probability. The literature is quite vast and the reader is referred to the sample of papers [3], [6], [18], [29], [31], [56], [58] and the references cited therein. The construction of such models involve the (unique) non-trivial bicharacter of $\mathbb{Z}_2$, which determines the Fermi commutation rules between *even* and *odd* elements. A first approach was shortly presented in [49], in which also the completion w.r.t. the spatial C^*-norm is considered. The maximal C^*-norm is briefly treated in [84]. To the knowledge of the authors, a systematic investigation of (abstract) Fermi C^*-systems started in [17], and continued in [31], where several aspects of the structure of *symmetric* states on the infinite chain of Fermi C^*-tensor products of a single algebra (including a quantum version of the celebrated De Finetti theorem) are thoroughly analysed following the lines of the seminal paper [76] (which deals with the infinite minimal C^*-tensor product) and of [18] (which addresses the CAR algebra on the chain $\mathbb{N}$).

In light of the previous considerations, a constructive approach to group-twisted tensor products, firstly from a purely algebraic point of view and secondly at a topological level, appears logical to face. Very recently, this investigation was carried out in [57], [70] by using the general structure of locally compact *quantum* groups and bicharacters on them. In the first one, a spatial construction is directly given by exploiting representations of the marginal algebras, without further investigating the minimality of the involved norm among all the admissible C^*-constructions. In the second paper, the maximal/universal counterpart is provided. Though rather general, these models are restricted to only two of all possible C^*-completions. Furthermore, they do not seem to lend themselves to easy and applicable computations, unless one limits himself to the setting of (classical) abelian groups, acting on the marginal algebras and giving rise to two C^*-dynamical systems. Surprisingly, this limitation is not so strong: as pointed out in Section 6.2 in [57] (see also our Remark II.7.6), the involved twisting always descends to the abelianization of a group in a canonical way. To take advantage of abstract Fourier analysis, we will also suppose that the acting groups are *compact*. Compactness will guarantee the existence of a faithful expectation onto the subalgebra of points fixed by the action, thus translating a C^*-dynamical system into a *topological grading* over the dual group (see Proposition II.4.3).

To summarize, the state of the art of the purely mathematical aspects of twisted C^*-tensor products is the following. The simplest one is the usual (i.e. non-twisted) tensor product where either one of the marginal actions or the bicharacter is trivial. Even in this apparently simple model, there are technical difficulties about C^*-completions, see *e.g.* [104]. The next step is turning to the simplest non-trivial examples: Fermi systems. In this situation, the involved groups establishing the grading on the marginal algebras are both $\mathbb{Z}_2$, and the bicharacter is the one establishing the CAR's between odd and even elements. Even in this case, there are just partial results about the topological aspects, essentially contained in [84], [17], [31], [49]. Another widely investigated model, which falls into the class of twisted C^*-tensor product, is the so-called noncommutative 2-torus, or rotation C^*-algebra. Here, the marginal C^*-algebras both coincide with the continuous functions on the torus $\mathbb{T}$, on which $\mathbb{T}$ itself acts transitively by rotations, and the bicharacter is $u(m,n) = e^{i2\pi\vartheta mn}$, with $m,n \in \widehat{\mathbb{T}} \cong \mathbb{Z}$ and $\vartheta \in (0,1)$. The case of irrational ϑ is of particular interest, even though also the rational case has many interesting features. The reader is referred to [85] and the references cited therein for further details. In view of various potential physical applications, we also mention [35] for a recent construction of type III representations, where Tomita modular data play a crucial role. Since the action of $\mathbb{T}$ on $\mathcal{C}(\mathbb{T})$ by rotations is ergodic, the C^*-rotation algebra has always a faithful state for each angle $\vartheta \in (0,1)$, indeed a trace (*e.g.* Proposition II.11.6), and thus the min and max-norm must coincide. This means that, apart from the (possible, but not yet clarified) existence of non-compatible norms, there are no problems on C^*-completions of the rotation algebra.

Our approach will be entirely constructive, hence more suited to applications. After some basic notation fixed in Section II.2, Section II.3 illustrates preliminary, well-established facts on involutive algebras, with a particular focus on *algebraically bounded* algebras, the ones whose convex cone generated by elements of the form a^*a forces all the algebra representations on some inner product space to act via bounded operators (hence, extendable to the Hilbert completion). This notion will be of use in the sequel, since both the algebraic layer of a graded C^*-algebra and the algebraic twisted tensor product are instances of bounded *-algebras (Proposition II.5.2, Proposition II.10.1). In Section II.4, we briefly mention the theoretical substrate of Fourier analysis necessary in the following, gradually narrowing the discussion to C^*-systems $(\mathfrak{A}, G, \alpha)$ based on a unital C^*-algebra $\mathfrak{A}$ and a pointwise norm-continuous action $G \overset{\alpha}{\curvearrowright} \mathfrak{A}$ of a compact group G on $\mathfrak{A}$, yielding a faithful expectation $E_G \colon \mathfrak{A} \twoheadrightarrow \mathfrak{A}^G$ onto the fixed point subalgebra $\mathfrak{A}^G$. This framework admits three pre-Hilbert module interpretations and, given any state $\varphi \in \mathcal{S}(\mathfrak{A})$, it allows to explicitly write the GNS triplets of the restriction $\varphi|_{\mathfrak{A}^G}$ $\mathfrak{A}^G$ and the E_G-pullback $\varphi \circ E_G$, by exploiting the Stinespring dilation of completely positive maps. The description is even clearer when G is abelian, hence admitting a dual group $\widehat{G}$ (necessarily, discrete) coacting in a C^*-algebraic sense on $\mathfrak{A}$ and grading it as an inner direct sum of spectral subspaces $\mathfrak{A}_\sigma$. The algebraic layer $\mathfrak{A}_o := \dotplus_{\sigma\in\widehat{G}} \mathfrak{A}_\sigma$, consisting of the grading homogeneous elements, is dense in $\mathfrak{A}$ and its fundamental properties, especially concerning its representations and states, are discussed in Section II.5. To achieve the construction of twisted tensor products an ingredient is left: group bicharacters. They are treated in Section II.6, which also contains a table of all the possible bicharacters on notable discrete abelian groups. After this preparation, in Section II.7 we construct the algebraic twisted tensor product of two C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ (the former graded by $\widehat{G}$, the latter by $\widehat{H}$) from a fixed bicharacter $u : \widehat{G} \times \widehat{H} \to \mathbb{T}$ which determines the commutation rules between homogeneous elements of $\mathfrak{A}_o$ and $\mathfrak{B}_o$. By such a commutation rule, it is also possible to deduce a *-operation which is compatible with the product, making $\mathfrak{A}_o \odot \mathfrak{B}_o$ a full-fledged involutive algebra denoted by $\mathfrak{A}_o \circledast \mathfrak{B}_o$. To be more specific, for the product and the adjoint operation, we set

$$(a \odot b)(A \odot B) := \overline{u(\partial A, \partial b)} aA \odot bB \quad (a \odot b)^* := \overline{u(\partial a, \partial b)} a^* \odot b^*$$

where ∂a is the degree of $a \in \mathfrak{A}_o$, and similarly for $A \in \mathfrak{A}_o, b, B \in \mathfrak{B}_o$. We also mod out the "inessential" parts of both the grading and the twisting, thus producing a sort of non-degenerate twisted product. The section ends with an analysis of the *factoring-out* mappings (see (II.1)) in this new setting. The three next sections, Section II.8, Section II.9 and Section II.10 deals with representations of $\mathfrak{A}_o \circledast \mathfrak{B}_o$, firstly establishing products of marginal representations, then discussing the Gel'fand-Neumark-Segal (GNS) representation induced by product states (where at least one of the marginal state must be invariant under the action of the corresponding group), and lastly proving that *every* representation of $\mathfrak{A}_o \,\textcircled{a}\, \mathfrak{B}_o$ must consist of bounded operators, thus permitting a logical investigation of C^*-completions. In Section II.9, we also point out a mistake in an equality in the proof of Proposition 7.1 in [17] which is circumvented by a our (hopefully, better) proof. In this occasion, the notion of *compatibility* and (*sub*)*-cross* properties of a C^*-norm on $\mathfrak{A}_o \,\textcircled{a}\, \mathfrak{B}_o$ are introduced. Precisely, a C^*-norm γ is compatible if the product action of $G \times H$ consists of γ-isometric automorphisms of $\mathfrak{A}_o \,\textcircled{a}\, \mathfrak{B}_o$, hence extending to a well-defined action on the γ-completion. Our definition differs from the one given in [19] and seems more correct, even if more restrictive: on the one hand, there exist norms which fits the definition in [19] but not ours (see Section II.13); on the other hand, our definition allows to correct the proof in [19] of the minimality of the spatial C^*-norm among all the compatible ones (see Section II.12). The cross property is defined in a stricter way than the one given in [19], as well (all the proofs in there still perfectly work with our definition). Our choice, this time, is not aimed to correct a mistake, but to guarantee the isometric embedding of the marginal algebras into the completion (see Proposition II.11.5). Maximal and minimal C^*-norms are described in Section II.11 from the class of all representation of $\mathfrak{A}_o \,\textcircled{a}\, \mathfrak{B}_o$ and the ones induced by products of invariant states, respectively. After showing that the min-norm is the smallest among all the compatible C^*-norms in Section II.12, we devote Section II.13 to the construction of a non-compatible norm, taking advantage of a pedagogical example due to S. Wassermann ([79]) in 1975. Another more recent example, due to Accardi, Fidaleo and Mukhamedov ([1]), is illustrated to show a crucial mistake in [19], responsible for leading to a wrong proof of the minimality of the min-norm. Section II.14 contains some useful characterizations of the max-norm in parallel to Section II.15 in which there are relevant characterizations of the min-norm, including its *spatiality* (Proposition II.15.1 and Remark II.15.2). We also provide the topological aspects of the factoring-out maps in relation to the max and min-norms, as well as of the "non-degeneracy" of a twisted product in Section II.16. When one of the fixed point subalgebras, $\mathfrak{A}^G$ and $\mathfrak{B}^G$, is nuclear, there exists a unique compatible C^*-norm on $\mathfrak{A}_o \,\textcircled{a}\, \mathfrak{B}_o$: this is the object of Section II.17. Lastly, Section II.18 is devoted to a generalization of the Klein-Jordan-Wigner transformation, initially aimed to realize a *-isomorphism between the Fermi (i.e. $\mathbb{Z}_2$-twisted) C^*-tensor product $\mathfrak{A} \,\textcircled{F}_{\min}\, \mathfrak{B}$ and the usual non-twisted one $\mathfrak{A} \otimes_{\min} \mathfrak{B}$, provided that at least one of the parity automorphisms on the involved C^*-algebras is inner. It was designed for applications to quantum field theory to pass from operators enjoying CAR's to ones enjoying CCR's. It was outlined in [49] and reconsidered in [31] in a purely C^*-algebraic setting. Here (Theorem II.18.1), we are able to extend this result to general twisted products. Namely, for C^*-systems $(\mathfrak{A}, G, \alpha)$ and $(\mathfrak{B}, H, \beta)$ such that either α or β is inner, there is a *-isomorphism between $\mathfrak{A} \,\textcircled{a}_{\min}\, \mathfrak{B}$ and $\mathfrak{A} \otimes_{\min} \mathfrak{B}$ preserving the invariant product states and intertwining the corresponding actions of $G \times H$. It should be specified that a kind of Klein transformation is actually outlined (even if not named) in [57] in their spatial description of the twisted tensor product, though without showing the last two properties just mentioned.

We end the present introduction with some potential applications. A relevant case of interest in quantum physics involves *anyons*. Anyons are quasi-particles living in low-dimentional space-time manifolds, and obeying to the braid statistics. Fractional anyons (detected by two experiments in 2020) were first theorised by F. Wilczek (cf. [80]). It is argued that they play a role in the fractional quantum Hall effect. It is also conjectured that they can have a role in quantum computing. The reader is referred to the review-paper [75], and [41] for a purely algebraic analysis finalized to applications in quantum probability. Concerning the appearance of anyons in algebraic quantum field theory, the reader is also referred to [37]. The particular statistics of these quasi-particles implies the *any*-commutation relation in the sense that, if one exchanges (i.e. commutes) operators with prefixed anyonic degree, any phase-factor might appear. From that, the statistics of such quasi-particles is named *anyonic statistics*. If, on the one hand, there are hundreds of papers studying the physics of such objects, on the other hand only few deal with the mathematical framework. By the above considerations, for the abstract construction of anyonic systems, we could argue that the natural candidate for the bicharacter is $u_{\text{any}}(x,y)=e^{i2\pi xy}$, $x,y\in\mathbb{R}/\mathbb{Z}$. Since the grade is induced by a compact group, which in this case should be the dual of the discrete circle $\mathbb{T}$, the involved C^*-systems would be of the form $(\mathfrak{A},\mathbf{Bohr}(\mathbb{Z}),\alpha)$ where $\mathbf{Bohr}(\mathbb{Z})$ is the Bohr compactification of $\mathbb{Z}$. We also mention the so-called *p-anyons* for which the involved dual groups are $\mathbb{Z}_p$.[1] Also non-abelian anyons, including the so-called *plektons*, have been introduced and studied for potential physical applications. In our framework, they might be achieved when the involved groups are still compact but not abelian, and thus the dual object would be merely a sort of discrete quantum group. In order to go beyond the model in [57] in the case of compact non-abelian groups, (then providing a model for such non-abelian anyons) we might take advantage of the harmonic analysis described in [23], and possibly give a more suitable definition for the commutation relations. We hope to treat this general framework somewhere else. We end by mentioning the adele construction of the dual of the discrete group $\mathbb{Q}$ of rational numbers under addition (or also of positive rationals $\mathbb{Q}_+$ under multiplication). The various twisted C^*-tensor products for which at least one of the involved compact groups is $\widehat{\mathbb{Q}}$ might have natural applications in Number Theory as well. The whole chapter is the content of a paper by Fidaleo F. and Vincenzi E., submitted to a journal and currently under review.

II.2. Preliminaries

Any topological space will be tacitly assumed to be Hausdorff. In particular, the topology of a (locally) compact space/group is assumed to be automatically Hausdorff.
If a group G *acts* on a space X through maps $X\ni x\mapsto gx\in X$ (see below for some standard useful cases), the *orbit* of an element x is denoted by $O_x\equiv G\cdot x:=\{g\cdot x: g\in G\}\subset X$. A point-space X on which a group G is acting is called a *G-space*, and such an action is schematically denoted by $G\curvearrowright X$.
For a vector space V, if not otherwise stated, we tacitly suppose that it is built on the field of complex numbers. Its algebraic dual, namely the linear space of all linear complex-valued functionals defined on V, is denoted by V'. If V is a topological vector space, its topological dual, i.e. the continuous elements in V', is denoted by V^*. In this framework, if $S\subseteq V$, $[S]:=\overline{\operatorname{span}_{\mathbb{C}} S}$ and S is *total* if $[S]=V$.
On a complex pre-Hilbert space, the inner product is supposed to be linear in the 1st variable and anti-linear in the 2nd. Let $\mathcal{H}$ be an, always complex, Hilbert space. With $\mathcal{B}(\mathcal{H})$, we denote the W^*-algebra of all bounded operators acting on $\mathcal{H}$. The identity $\mathbb{1}_{\mathcal{B}(\mathcal{H})}$ coincides with the identity operator $I_{\mathcal{H}}$, and is simply denoted by I when this causes no confusion. We denote by $\mathcal{K}(\mathcal{H})$ the norm-closed, two-sided *-ideal of all *compact* operators, by $\mathcal{B}_1(\mathcal{H})\subset\mathcal{K}(\mathcal{H})$ the (not necessarily norm-closed) two-sided *-ideal of all *trace-class* operators, and by $\mathcal{U}(\mathcal{H})$ the norm-closed group of all *unitary* operators acting on $\mathcal{H}$. For Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, their complete tensor product is again a Hilbert space denoted by $\mathcal{H}\otimes\mathcal{K}$. For $\xi\in\mathcal{H}$, the corresponding *vector functional* on $\mathcal{B}(\mathcal{H})$ is defined as $\omega_\xi:=\langle\cdot\,\xi,\xi\rangle$.

Let X and Y be two linear spaces. With $X\dotplus Y$ and $X\odot Y$ we denote their (outer) direct sum and tensor product, respectively. We also consider an arbitrary collection $(X_\iota)_\iota$ of linear spaces. Their outer direct sum consists of all nets $\dotplus_\iota X_\iota:=\big\{\mathbf{x}=(x_\iota)_\iota\colon x_\iota=0 \text{ but a finite number of indices}\big\}$, where the sum and product-by-scalars are component-wise defined as $\mathbf{x}+\mathbf{y}:=(x_\iota+y_\iota)_\iota$ and

[1] This model for $p=1,2,\dots$, is treated in a purely algebraic way in [41], after noticing that the 1-anyons are exactly the bosons and the 2-anyons are the fermions, then providing the usual and the Fermi tensor product, respectively.

$c\mathbf{x} := (cx_\iota)_\iota$ $(c \in \mathbb{C})$, respectively. If $X_\iota \subset X$, then it is also possible to define the inner direct sum $\dotplus_\iota X_\iota \subset X$, provided that $\iota \neq \kappa \Rightarrow X_\iota \cap X_\kappa = \{0\}$. Suppose that Z is another linear space. It is well known that

$$(X \odot Z) \dotplus (Y \odot Z) \cong (X \dotplus Y) \odot Z, \text{ and } (Z \odot X) \dotplus (Z \odot Y) \cong Z \odot (X \dotplus Y),$$

through the (*right* and *left*) *factoring-out maps* R and L defined on the elementary tensors as

$$\begin{aligned} R\colon (X \odot Z) \dotplus (Y \odot Z) &\to (X \dotplus Y) \odot Z \\ (x \odot z_1, y \odot z_2) &\mapsto (x,0) \odot z_1 + (0,y) \odot z_2\,, \\ L\colon (Z \odot X) \dotplus (Z \odot Y) &\to Z \odot (X \dotplus Y) \\ (z_1 \odot x, z_2 \odot y) &\mapsto z_1 \odot (x,0) + z_2 \odot (0,y)\,. \end{aligned} \tag{II.1}$$

If in addition, $\mathfrak{A}$ and $\mathfrak{B}$ are involutive algebras, then $\mathfrak{A} \otimes \mathfrak{B}$ will denote the algebraic tensor product $\mathfrak{A} \odot \mathfrak{B}$ equipped with the usual product and involution given on the simple tensors by

$$(a_1 \otimes b_1)\cdot(a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2, \quad (a_1 \otimes b_1)^\dagger = a_1^* \otimes b_1^*\,,$$

for all $a_1, a_2 \in \mathfrak{A}, b_1, b_2 \in \mathfrak{B}$. We are adopting the symbols "$\cdot$" and "$\dagger$" to denote the product and the involution in $\mathfrak{A} \otimes \mathfrak{B}$ just in order to distinguish them from the analogous operations (denoted by the simple juxtaposition and "$*$", respectively) on the twisted tensor product $\mathfrak{A}_o \mathbin{\textcircled{w}} \mathfrak{B}_o$ (and its natural completions), as we shall see. Lastly, we reserve the symbol "$\oplus$" for Banach direct sums and C^*-algebraic direct sum.

II.3. Representations of involutive algebras

Since we heavily deal with involutive algebras, often without any *a priori* assigned topology, we fix some basic notation. An *involutive* (or *-) algebra is a complex algebra A, always unital if not otherwise specified, equipped with an antilinear involution * such that $\mathbb{1}_A^* = \mathbb{1}_A$. As usual, A_{sa} denotes the Jordan real algebra consisting of the *selfadjoint* elements in A, i.e. $a \in A_{\mathrm{sa}}$ if and only if $a = a^*$. Particularly, $a \in A_{\mathrm{sa}}$ is (*algebraically*) *positive* if $a = z^* z$ for some $z \in A$. Furthermore, following the notation employed in [101], we denote by $\sum A^2$ the *cone generated* by the positive elements of A:

$$\sum A^2 := \left\{ \sum_{i=1}^n z_i^* z_i \colon z_i \in A, n \geq 1 \right\} \subset A\,.$$

Evidently, $\sum A^2$ is convex (or, equivalently, closed under addition), hence it induces a partial ordering on A, as customary: for $s, t \in A$, we write $s \leq_A t$ if $t - s \in \sum A^2$. Moreover, $\sum A^2$ is a *quadratic module* of A, that is $\mathbb{1}_A \in \sum A^2$ and $x^* s x \in \sum A^2$ for every $s \in \sum A^2$ and $x \in A$. Without loss of generality, we will only deal with non-degenerate representations of A acting on, not necessarily complete, complex inner product spaces $(\mathcal{H}_o, \langle \cdot, \cdot \rangle)$. Therefore, a *representation* of the involutive algebra A is merely a *-algebra homomorphism π from A to $\mathcal{L}(\mathcal{H}_o)$, the set of all $\mathbb{C}$-linear operators acting on $\mathcal{H}_o$, satisfying

$$\langle \pi(x)\xi, \eta \rangle = \langle \xi, \pi(x^*)\eta \rangle\,, \quad x \in A,\ \xi, \eta \in \mathcal{H}_o\,.$$

The representation π is *non-degenerate* if $\xi \in \mathcal{H}_o$, $\xi \perp \pi(A)\mathcal{H}_o \Rightarrow \xi = 0$. It is easy to see that this is equivalent to ask π to be unital, i.e. $\pi(\mathbb{1}_A) = I_{\mathcal{H}_o}$ (see [101], Lemma 4.9 (iii), p. 64).

Again from [101], A is *algebraically bounded* (simply shortened as "bounded„ in the sequel) if the quadratic module $\sum A^2$ is *Archimedean*, i.e. for each $a \in A$ there exists a positive constant $C_a > 0$ (only depending on a) s.t. $a^*a \leq_A C_a \mathbb{1}_A$. The reason of the term "bounded„ is given by the straightforward fact that every representation $\pi : A \to \mathcal{L}(\mathcal{H}_o)$ of a bounded *-algebra A has range $\pi(A)$ lying in $\mathcal{B}(\mathcal{H}_o)$, thus determining a representation $\overline{\pi}\colon A \to \mathcal{B}(\mathcal{H})$ acting on the completion $\mathcal{H} := \overline{\mathcal{H}_o}$ by bounded operators, as it will be explained in the following

Lemma II.3.1
Each representation π of a bounded *-algebra A on a pre-Hilbert space $\mathcal{H}_o$ uniquely extends to a representation acting by bounded operators on the Hilbert space completion $\mathcal{H} = \overline{\mathcal{H}_o}$. By setting $\overline{\pi}(a) := \overline{\pi(a)}$ $(a \in A)$, π induces a full-fledged representation $\overline{\pi}\colon A \to \mathcal{B}(\mathcal{H})$.

Proof.
Let π be a representation of A on a pre-Hilbert space $\mathcal{H}_o$. Then, for each $a \in A$, $\pi(a)$ is closable. We now show that it is even bounded, if A is a bounded *-algebra. By hypothesis, for every $a \in A$ there exists $C_a > 0$ such that $C_a \mathbb{1}_A - a^*a \in \sum A^2$, hence

$$0 \leq \langle \pi(C_a \mathbb{1}_A - a^*a)\xi, \xi\rangle_{\mathcal{H}_o} = C_a \|\xi\|^2_{\mathcal{H}_o} - \|\pi(a)\xi\|^2_{\mathcal{H}_o}, \quad \xi \in \mathcal{H}_o \,,$$

an thus $\pi(a)$ is a bounded operator on $\mathcal{H}_o$. It follows that

$$D\left(\overline{\pi(a)}\right) = \left\{ \xi \in \overline{\mathcal{H}_o}\colon \exists (\xi_n)_n \subset D(\pi(a)) \text{ s.t. } \begin{cases} \xi_n \to \xi \\ (\pi(a)\xi_n)_n \text{ Cauchy} \end{cases} \right\} = \overline{\mathcal{H}_o}\,,$$

and therefore $\overline{\pi(a)} \in \mathcal{B}(\overline{\mathcal{H}_o})$ by the Closed Graph Theorem. In particular, $\overline{\pi(a)}$ is the unique continuous extension of $\pi(a)$ to the Hilbert space $\overline{\mathcal{H}_o}$ and $\|\overline{\pi(a)}\|_{\mathcal{B}(\overline{\mathcal{H}_o})} = \|\pi(a)\|_{\mathcal{B}(\mathcal{H}_o)}$. We are left to show that the mapping $\overline{\pi}\colon A \to \mathcal{B}(\mathcal{H})$, $\overline{\pi}(a) := \overline{\pi(a)}$ defines a representation of A on $\mathcal{H} := \overline{\mathcal{H}_o}$.
For each $\xi, \eta \in \mathcal{H}$, choose two sequences $(\xi_n), (\eta_n) \subset \mathcal{H}_o$ converging to ξ and η, respectively. For the *-operation, by joint continuity of the inner product we get

$$\begin{aligned} \langle \overline{\pi}(a^*)\xi, \eta\rangle =& \langle \overline{\pi(a^*)}\xi, \eta\rangle = \lim_{m,n} \langle \pi(a^*)\xi_m, \eta_n\rangle = \lim_{m,n} \langle \xi_m, \pi(a)\eta_n\rangle \\ =& \langle \xi, \overline{\pi(a)}\eta\rangle = \langle \overline{\pi(a)}^*\xi, \eta\rangle = \langle \overline{\pi}(a)^*\xi, \eta\rangle, \quad a \in A. \end{aligned}$$

As concerns the product, first notice that the sequence $(\pi(b)\xi_n)_n \subset \mathcal{H}_o$ converges to $\overline{\pi(b)}\xi \equiv \overline{\pi}(b)\xi \in \mathcal{H}$. Therefore,

$$\begin{aligned} \overline{\pi}(a)\overline{\pi}(b)\xi =& \overline{\pi}(a)(\overline{\pi(b)}\xi) = \lim_n \overline{\pi}(a)(\pi(b)\xi_n) = \lim_n \overline{\pi(a)}(\pi(b)\xi_n) \\ =& \lim_n \pi(a)\pi(b)\xi_n = \lim_n \pi(ab)\xi_n = \overline{\pi(ab)}\xi = \overline{\pi}(ab)\xi, \quad a, b \in A. \end{aligned}$$ □

Clearly, every C^*-algebra is bounded. If A is a bounded *-algebra, let $\mathsf{Rep}(A)$ be the family of (unitary equivalence classes of) non-degenerate representations of A on some Hilbert space. We shall show soon (cf. Proposition II.10.1) that all the involutive algebras arising from twisted tensor products are bounded. We recall that an algebra-norm $\| \ \|$ on A is always supposed to satisfy $\|a\| = \|a^*\|$ for each $a \in A$, i.e. the involution * is norm isometric. Any C^*-*norm* on A evidently satisfies this property. An involutive algebra equipped with a C^*-norm $\| \ \|$ is named *pre-C^*-algebra*, seen as a dense *-subalgebra of its C^*-completion.
We also recall that, for a positive functional f on a normed *-algebra $(A, \| \ \|_A)$, f is $\| \ \|_A$-*weakly bounded on a subset* $\mathcal{X} \subset A$ if all mappings of the form

$$\mathcal{X} \ni x \mapsto f(a^*xa) \in [0, +\infty)\,, \quad a \in A\,,$$

are norm-continuous. Likewise, the representation $\pi\colon A \to \mathcal{L}(\mathcal{H}_o)$ of A on a pre-Hilbert space $\mathcal{H}_o$ is said to be *weakly continuous on a subset* $\mathcal{X} \subset A$ if all the linear forms

$$\mathcal{X} \ni x \mapsto \omega_\xi(x) := \langle \pi(x)\xi, \xi \rangle_{\mathcal{H}_o}\,, \quad \xi \in \mathcal{H}_o\,,$$

are norm-continuous. In both cases, if $\mathcal{X} = A$, we shall simply say “weakly continuous„/“weakly bounded„.

Interestingly, it can be shown that, when π is weakly continuous on $\mathcal{X} := A_{\mathrm{sa}}$, $\pi(A) \subset \mathcal{B}(\mathcal{H}_o)$ and then each operator $\pi(a)$ can be uniquely extended to a bounded operator $\overline{\pi(a)} \in \mathcal{B}(\mathcal{H})$ (see Theorem 25.5 in [106]), coinciding with its (unique) closed extension.

Instances of representations of a *-algebra A, ubiquitous in literature, are the ones induced by (algebraically) positive functionals. If $f\colon A \to \mathbb{C}$ is a positive functional, let $\langle a, b\rangle_f := f(b^*a)$ $(a, b \in A)$ the positive semidefinite, sesquilinear form associated to f. Evidently, it satisfies

- $\langle ab, c\rangle_f = \langle b, a^*c\rangle_f$ for $a, b, c \in A$
- $\langle a, b\rangle_f = \overline{\langle b, a\rangle_f}$ for $a, b \in A$ (Hermitianness of $\langle\cdot,\cdot\rangle_f$)
- $|\langle a, b\rangle|^2 \le \langle a, a\rangle_f \langle b, b\rangle_f$ for $a, b \in A$ (Cauchy-Bunyakovsky-Schwarz inequality)

Denote by $\mathfrak{n}_f := \{a \in A\colon \langle a, b\rangle_f = 0,\ b \in A\} = \{a \in A\colon f(a^*a) = 0\}$ the *left kernel* of the functional f, a left (in general, not *-closed) ideal of A. The quotient space $\mathcal{H}_{o,f} := A/\mathfrak{n}_f = \{a_f\colon a \in A\}$ is then equipped with a well-defined inner product

$$\langle a_f, b_f\rangle_{\mathcal{H}_{o,f}} := \langle a, b\rangle_f = f(b^*a)\,, \quad a_f, b_f \in \mathcal{H}_{o,f}\,,$$

thence getting a structure of pre-Hilbert space, upon which the *left multiplication* representation of A

$$\pi_f(a)b_f := (ab)_f\,, \quad a, b \in A\,,$$

acts via linear operators, with *algebraically cyclic* vector $\xi_f := \big(\mathbb{1}_A\big)_f$: $\pi_f(A)\xi_f = \mathcal{H}_{o,f}$. The explicit relation between f and the associated representation π_f is then, as is known, given by

$$f(a) = \langle \pi_f(a)\xi_f, \xi_f\rangle_{\mathcal{H}_{o,f}}, \quad a \in A\,.$$

If $\pi_f(A) \subset \mathcal{B}(\mathcal{H}_{f,o})$, π_f is said to be the *GNS* (Gel'fand, Neumark and Segal) *representation* associated to f. As seen in Lemma II.3.1, it yields a representation $\overline{\pi_f}\colon A \to \mathcal{B}(\mathcal{H}_f)$, where $\mathcal{H}_f := \overline{\mathcal{H}_{o,f}}$. When A is equipped with an algebra norm, one can straightforwardly characterize norm-continuity of a positive functional f via the associated representation.

Proposition II.3.2
Let A be a normed, unital *-algebra, and f a non-zero, algebraically positive, linear functional on A. Under the notation of Lemma II.3.1, the following are equivalent:

(1) $f\colon A \to \mathbb{C}$ is bounded;

(2) $\pi_f\colon A \to \mathcal{L}(\mathcal{H}_{f,o})$ acts by bounded operators, and $\overline{\pi_f}\colon A \to \mathcal{B}(\mathcal{H}_f)$ is bounded;

(3) $\pi_f\colon A \to \mathcal{L}(\mathcal{H}_{f,o})$ acts by bounded operators, and $\overline{\pi_f}\colon A \to \mathcal{B}(\mathcal{H}_f)$ is weakly continuous.

In the event that $\|\ \|$ is a C^*-norm and $\mathfrak{A} := \overline{A}^{\|\ \|}$ (with positive cone $\mathfrak{A}_+ := \{z^*z\colon z \in \mathfrak{A}\}$), the previous three properties are equivalent to each of the following:

(4) $f(\mathfrak{A}_+ \cap A) \subset [0, +\infty)$;

(5) $\pi_f(\mathfrak{A}_+ \cap A) \subset \mathcal{B}(\mathcal{H}_f)_+$.

Proof.
We will show the implication scheme (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3).
(1) $\Rightarrow$ (2): since f is bounded, for each vector $\xi := \pi_f(b)\xi_f$, we get

$$|\omega_\xi(x)| = |\langle \pi_f(b^*xb)\xi_f, \xi_f\rangle_{\mathcal{H}_{o,f}}| = |f(b^*xb)| \leq C\|b\|_A^2\|x\|_A\,, \quad x \in A_{\mathrm{sa}}\,.$$

i.e. π is weakly continuous on the subset A_{sa}. By Theorem 25.5 in [106], we deduce that $\pi_f(A) \subset \mathcal{B}(\mathcal{H}_{o,f})$. Lastly, since f admits a unique algebraically positive continuation $\overline{f}$ to the Banach *-algebra $\overline{A}^{\|\ \|}$ (with automatically contractive GNS representation $\pi_{\overline{f}}$ on the Hilbert space $\mathcal{H}_{\overline{f}}$, by Corollary 25.17 in [106]), $\overline{\pi_f}\colon A \to \mathcal{B}(\mathcal{H}_f)$ and $\pi_{\overline{f}}\big|_A\colon A \to \mathcal{B}(\mathcal{H}_{\overline{f}})$ are unitarily equivalent representations of A (see *e.g.* [101], Theorem 4.41, pp. 80-81; here, the unitality requirement on A is used). It follows that $\overline{\pi_f}$ is bounded (even contractive).
(2) $\Rightarrow$ (1): by boundedness of $\overline{\pi_f}$,

$$|f(x)| \leq \|\overline{\pi_f}(x)\|\|\xi_f\|^2 \leq C\|\xi_f\|^2\|x\|_A$$

for every $x \in A$, whence f is bounded.
(2) $\Rightarrow$ (3) is obvious.
(3) $\Rightarrow$ (2): we shall apply the uniform boundedness principle twice. Using the polarization identity on $\mathcal{H}_f$ and the decomposition of elements of A in their respective real and imaginary parts, for every fixed $\xi, \eta \in \mathcal{H}_f$ the linear functionals $A \ni a \mapsto \langle\overline{\pi_f(a)}\xi, \eta\rangle_{\mathcal{H}_f}$ are bounded. By the Riesz representation theorem, this is equivalent to say that the ξ-section $S_\xi := \left\{\overline{\pi_f(a)}\xi\colon \|a\| \leq 1\right\} \subset \mathcal{H}_f$ is weakly bounded, that is $\phi(S_\xi)$ is bounded in $\mathbb{C}$ for every $\phi \in \mathcal{H}_f^*$. By the Banach-Steinhaus theorem, S_ξ is then bounded in $\mathcal{H}_f$, that is $\sup\limits_{\|a\|\leq 1}\left\|\overline{\pi_f(a)}\xi\right\| \leq C_\xi$ for every fixed $\xi \in \mathcal{H}_f$. Again applying Banach-Steinhaus, $\left(\overline{\pi_f(a)}\right)_{a\in A\colon \|a\|\leq 1} \subset \mathcal{B}(\mathcal{H}_f)$ is a bounded family of operators, that is $\overline{\pi_f}\colon A \to \mathcal{B}(\mathcal{H}_f)$ is bounded.
The last part of the lemma is easily implied by Propositions 2.1 (p. 10) and 2.11 (p. 16) in [96], since A is a particular instance of (dense) operator system in $\mathfrak{A}$. □

Remark II.3.3
Recall that $f\colon A \to \mathbb{C}$ in the previous lemma is bounded if and only if $\ker f$ is closed if and only if the graph of f is closed.

As for C^*-algebras, a *state* $\varphi \in \mathcal{S}(A)$ on a unital *-algebra is a (algebraically) positive unital functional on A, that is $\varphi\left(\sum A^2\right) \subseteq [0, +\infty)$ and $\varphi(\mathbb{1}_A) = 1$. We shall say that a family of states $\mathcal{S} \subset \mathcal{S}(A)$ *separates the points* of A (or that it is *point separating* for A) if the condition "$\varphi(a^*a) = 0$ for each $\varphi \in \mathcal{S}$„ implies that $a = 0$. If $\varphi \in \mathcal{S}(A)$ is such that $\mathcal{S} = \{\varphi\}$ is point separating for A, then φ is said to be *faithful*.

II.4. Abstract Fourier analysis on C^*-algebras

We outline here the structure of a *graded* C^*-algebra arising from the action of a compact (abelian) group. First of all, let us give some basic facts, useful for the upcoming theory. If $\mathfrak{B} \subset \mathfrak{A}$ is an inclusion of C^*-algebras, a *projection* of $\mathfrak{A}$ onto $\mathfrak{B}$ is a surjective linear map $E\colon \mathfrak{A} \twoheadrightarrow \mathfrak{B}$ s.t. $E(b) = b$ for each $b \in \mathfrak{B}$. If, in addition, E is contractive, it is said to be a *(conditional) expectation* of $\mathfrak{A}$ onto $\mathfrak{B}$. It is well known that an expectation E is automatically completely positive and a $\mathfrak{B}$-bimodule map, i.e. $E(b_1ab_2) = b_1E(a)b_2$ ($a \in \mathfrak{A}, b_1, b_2 \in \mathfrak{B}$).

Trivially, $\mathfrak{B} \subset \mathcal{M}_E$, where $\mathcal{M}_E := \{a \in \mathfrak{A} \colon E(a^*a) = E(a)^*E(a), E(aa^*) = E(a)E(a)^*\}$ is the *multiplicative domain* of E, a C^*-subalgebra of $\mathfrak{A}$. Notice that if $\mathfrak{B} \subset \mathfrak{A}$ have a common unity and $E \colon \mathfrak{A} \twoheadrightarrow \mathfrak{B}$ is a unital projection, then it is an expectation iff it is positive.

A faithful expectation E of $\mathfrak{A}$ onto $\mathfrak{B}$ (i.e. one for which $\mathfrak{A}_+ \cap \ker E = \{0\}$) gives $\mathfrak{A}$ a structure of *pre-Hilbert* **-bimodule* over $\mathfrak{B}$, i.e. of *-closed $\mathfrak{B}$-bimodule equipped with a bilinear $\mathfrak{B}$-valued map

$$\langle \cdot, \cdot \rangle_E \colon \mathfrak{A} \times \mathfrak{A} \to \mathfrak{B}$$
$$(a_1, a_2) \mapsto \langle a_1, a_2 \rangle_E := E(a_1 a_2)$$

satisfying

- $\langle ba_1, a_2 \rangle_E = b\langle a_1, a_2 \rangle_E$, $\langle a_1, a_2 b \rangle_E = \langle a_1, a_2 \rangle_E b$ and $\langle a_1 b, a_2 \rangle_E = \langle a_1, ba_2 \rangle_E$, $a_1, a_2 \in \mathfrak{A}, b \in \mathfrak{B}$ ($\mathfrak{B}$-bilinearity)
- $\langle a_1, a_2 \rangle_E^* = \langle a_2^*, a_1^* \rangle_E$, $a_1, a_2 \in \mathfrak{A}$
- $\langle a, a^* \rangle_E \in \mathfrak{A}_+$, $a \in \mathfrak{A}$
- $\langle a, a^* \rangle_E = 0$ if and only if $a = 0$

The map $\langle \cdot, \cdot \rangle_E$ also induces two sesquilinear, positive definite, Hermitian, $\mathfrak{B}$-valued maps $\langle a_1, a_2 \rangle_l := \langle a_1, a_2^* \rangle_E = E(a_1 a_2^*)$, $\langle a_1, a_2 \rangle_r := \langle a_1^*, a_2 \rangle_E = E(a_1^* a_2)$, and hence three norms $\|a\|_l := \sqrt{\langle a, a \rangle_l} = \sqrt{E(aa^*)}$, $\|a\|_r := \sqrt{\langle a, a \rangle_r} = \sqrt{E(a^*a)}$ and $\|a\|_m := \|a\|_l \vee \|a\|_r = \|a\|_l \vee \|a^*\|_l$. If $\| \, \|_m$ is complete (i.e. $\| \, \|_m$-Cauchy sequences converge), then $\mathfrak{A}$ is said to be a *Hilbert* **-bimodule* over $\mathfrak{B}$. We remark that:

(1) $(\mathfrak{A}, \langle \cdot, \cdot \rangle_l)$ is a left pre-Hilbert $\mathfrak{B}$-module, whereas $(\mathfrak{A}, \langle \cdot, \cdot \rangle_r)$ a right one

(2) the involution * is $\| \, \|_m$-isometric

(3) $\|E(a)\|_{\mathfrak{A}} \leq \|a\|_l \wedge \|a\|_r \leq \|a\|_m \leq \|a\|_{\mathfrak{A}}$, $a \in \mathfrak{A}$; in particular, the metric topology induced by $\| \, \|_m$ is coarser than the one associated to $\| \, \|_{\mathfrak{A}}$

We are particularly interested in projections coming from C^*-systems. A C^*-system (or briefly, C^*-system) is a triplet $(\mathfrak{A}, G, \alpha)$ consisting of a (unital) C^*-algebra, a (Hausdorff) topological group G and a strongly continuous (that is, continuous in the point-norm topology) action $G \overset{\alpha}{\curvearrowright} \mathfrak{A}$, i.e. a representation $G \ni g \mapsto \alpha_g \in \mathsf{Aut}(\mathfrak{A})$ of G via *-automorphisms of $\mathfrak{A}$ s.t. for each fixed $a \in \mathfrak{A}$, the mapping $g \mapsto \alpha_g(a)$ is norm-continuous. Sometimes, for $a \in \mathfrak{A}$ and $g \in G$, we will also write $g(a)$ in place of $\alpha_g(a)$, in order not to overload the notation. Also, if there is no matter of confusion, we will omit to indicate the symbol α where possible. For instance, the fixed point C^*-subalgebra made of all the G-stable elements will be simply denoted by $\mathfrak{A}^G := \{a \in \mathfrak{A} \colon g(a) = a, \, g \in G\}$. Apparently, $\mathbb{C}\mathbb{1}_{\mathfrak{A}} \subseteq \mathfrak{A}^G$. If $\mathfrak{A}^G = \mathbb{C}\mathbb{1}_{\mathfrak{A}}$, the C^*-system $(\mathfrak{A}, G, \alpha)$, or the action α, is said to be *ergodic*. If $\mathfrak{A}$ is abelian, even non-unital, by Gel'fand-Neumark theorem $\mathfrak{A} \cong \mathcal{C}_o(\Omega_{\mathfrak{A}})$ where $\Omega_{\mathfrak{A}} := \mathsf{Hom}(\mathfrak{A}, \mathbb{C})$ is a locally compact, Hausdorff space when endowed with the topology of pointwise convergence, so that $\mathsf{Aut}(\mathfrak{A}) = \mathsf{Aut}(\mathcal{C}_o(\Omega_{\mathfrak{A}})) = \{\phi_* \colon \phi \in \mathsf{Homeo}(\Omega_{\mathfrak{A}})\}$ ($\phi_*(f) := f \circ \phi^{-1}$ for $f \in \mathcal{C}_o(\Omega_{\mathfrak{A}})$ is the *pushforward* of ϕ). It follows that, when $\mathfrak{A}$ is abelian, there exists a 1-1 correspondence between C^*-systems $(\mathfrak{A}, G, \alpha)$ and classical dynamical system $(\Omega_{\mathfrak{A}}, G^{\mathrm{op}}, \alpha_*)$.

Let $(\mathfrak{A}, G, \alpha)$ be a C^*-system. The pullback of the strongly continuous action $G \overset{\alpha}{\curvearrowright} \mathfrak{A}$ is a weakly-* continuous action $G \overset{\alpha^*}{\curvearrowright} \mathcal{S}(\mathfrak{A})$, and $\mathcal{S}_G(\mathfrak{A}) := \mathrm{Fix}(\alpha^*) = \{\varphi \in \mathcal{S}(\mathfrak{A}) \colon \varphi \circ \alpha_g = \varphi, \, g \in G\}$

is the convex space of the G-*invariant* states. Since we always deal with unital C^*-algebras, $\mathcal{S}_G(\mathfrak{A})$ is also weakly-* compact. Its extremal boundary

$$\mathcal{E}_G(\mathfrak{A}) := \mathrm{Ext}(\mathcal{S}_G(\mathfrak{A})) = \{\varphi \in \mathcal{S}_G(\mathfrak{A})\colon \varphi = t\varphi_1 + (1-t)\varphi_2, t \in (0,1), \varphi_i \in \mathcal{S}_G(\mathfrak{A}) \Rightarrow \varphi_1 = \varphi_2 = \varphi\}$$

consists of the so-called G-*ergodic* (or simply, *ergodic*) states. We will see soon that the choice of this term is linked to the relative "purity„ (viz "integral indecomposability„) of elements in $\mathcal{E}_G(\mathfrak{A})$ among all the G-invariant ones. By Krejn-Mil'man theorem, $\mathcal{E}_G(\mathfrak{A}) \neq \varnothing$ and $\mathcal{S}_G(\mathfrak{A}) = \overline{\mathrm{co}}^{w^*}(\mathcal{E}_G(\mathfrak{A}))$, the weak-* closure of the convex hull of $\mathcal{E}_G(\mathfrak{A})$. For each $\varphi \in \mathcal{S}_G(\mathfrak{A})$, there exists a GNS *covariant* representation $(\mathcal{H}_\varphi, \pi_\varphi, U_\varphi, \xi_\varphi)$ of $\mathfrak{A}$, i.e. the triple $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$ is the unique (up to unitary equivalence) GNS representation associated to φ and $U_\varphi\colon G \to \mathcal{U}(\mathcal{H}_\varphi)$ is a strongly continuous unitary representation of G on $\mathcal{H}_\varphi$ satisfying

$$\pi_\varphi(\alpha_g(a)) = \mathrm{ad}_{U_\varphi(g)}(\pi_\varphi(a)) = U_\varphi(g)\pi_\varphi(a)U_\varphi(g)^*, \quad g \in G,\ a \in \mathfrak{A}$$

$$U_\varphi(g)\xi_\varphi = \xi_\varphi, \quad g \in G$$

Let $\mathcal{H}_\varphi^G := \{\xi \in \mathcal{H}_\varphi\colon U_\varphi(g)\xi = \xi,\ g \in G\}$ and $P_\varphi\colon \mathcal{H}_\varphi \twoheadrightarrow \mathcal{H}_\varphi^G$ the associated orthogonal projection. Notice that the inner action $G \overset{\mathrm{ad}_{U_\varphi}}{\curvearrowright} \mathcal{B}(\mathcal{H}_\varphi)$ is pointwise σ-weakly (equivalently, pointwise weakly or pointwise strongly) continuous, therefore not only the C^*-algebra $\pi_\varphi(\mathfrak{A})$, but also the generated GNS von Neumann algebra $\pi_\varphi(\mathfrak{A})''$ is left invariant by ad_{U_φ}. Let $\mathsf{Rep}(\mathfrak{A})$ be the family of (unitary equivalence classes of) non-degenerate representations of $\mathfrak{A}$ on some Hilbert space. Similarly, let $\mathsf{URep}(G)$ be the family of (unitary equivalence classes of) strongly continuous unitary representations of G on Hilbert spaces. The previous discussion says that $\varphi \in \mathcal{S}_G(\mathfrak{A})$ induces $(\pi_\varphi, U_\varphi) \in \mathsf{Cov}(\mathfrak{A}, G, \alpha)$, where

$$\mathsf{Cov}(\mathfrak{A}, G, \alpha) := \{(\pi, U_\pi)\colon \pi \in \mathsf{Rep}(\mathfrak{A}), U_\pi \in \mathsf{URep}(G),\ \pi \circ \alpha_g = \mathrm{ad}_{U(g)},\ g \in G\}\,.$$

From now on, let us suppose G compact. Then, there exists a unital, faithful, G-invariant expectation $E_G\colon \mathfrak{A} \twoheadrightarrow \mathfrak{A}^G$ onto the fixed point subalgebra $\mathfrak{A}^G$ given by the Bochner integral

$$E_G(a) := \int_G \alpha_g(a)\mathrm{d}g, \quad a \in \mathfrak{A} \tag{II.2}$$

where $\mathrm{d}g$ denotes the probability Haar measure on G (that is, the unique inner-outer regular, Borel probability measure on G which is invariant under G-translations). It is clear that if $(\mathfrak{A}, G, \alpha)$ and $(\mathfrak{B}, H, \beta)$ are two C^*-systems and $\pi \in \mathsf{Hom}(\mathfrak{A}, \mathfrak{B})$ is equivariant (i.e. $\pi \circ \alpha = \beta \circ \pi$), then faithfulness of π on $\mathfrak{A}$ and on $\mathfrak{A}^G$ are equivalent, in which case $\pi \circ E_G$ is a faithful c.p.u. map (for a reference, see Section 4.5 in [88], p. 133). In view of the above discussion, the C^*-algebra $\mathfrak{A}$ inherits three pre-Hilbert module structures over its fixed point subalgebra $\mathfrak{A}^G$: the first of *-*bimodule* given by $\langle a_1, a_2\rangle_{E_G} := \int_G \alpha_g(a_1)\alpha_g(a_2)\mathrm{d}g$, the second of *left* module given by $\langle a_1, a_2\rangle_l := \int_G \alpha_g(a_1)\alpha_g(a_2)^*\mathrm{d}g$, the last of *right* module given by $\langle a_1, a_2\rangle_r := \int_G \alpha_g(a_1)^*\alpha_g(a_2)\mathrm{d}g$.

Coming back to G-invariant states, now we can write $\mathcal{S}_G(\mathfrak{A}) = E_G^{\mathrm{t}}\left(\mathcal{S}(\mathfrak{A}^G)\right)$ and the mappings

$$\begin{aligned} \mathcal{S}_G(\mathfrak{A}) &\to \mathcal{S}(\mathfrak{A}^G) \\ \varphi &\mapsto \varphi|_{\mathfrak{A}^G} \\ f \circ E_G &\leftarrow\!\shortmid f \end{aligned}$$

are affine, weakly-* bicontinuous homeomorphisms, one the inverse of the other. The ergodicity of the action α is then equivalent to requiring that the C^*-system $(\mathfrak{A}, G, \alpha)$ is *uniquely ergodic* (i.e.

$\mathcal{S}_G(\mathfrak{A})$ is a singleton, consisting of a unique G-invariant, faithful state ω_α s.t. $E_G(a) = \omega_\alpha(a)\mathbb{1}_{\mathfrak{A}}$, $a \in \mathfrak{A}$), see [30] and the references cited therein. Thanks to the faithfulness of E_G, one straightforwardly sees that $\mathcal{S}_G(\mathfrak{A})$ is point separating for $\mathfrak{A}$, and by an easy application of Krejn-Mil'man theorem the same holds for $\mathcal{E}_G(\mathfrak{A})$.

Moreover, given any state $\varphi \in \mathcal{S}(\mathfrak{A})$, we can always express the GNS representations of its restriction $\varphi|_{\mathfrak{A}^G} \in \mathcal{S}(\mathfrak{A}^G)$ and of its E_G-pullback $E_G^{\mathrm{t}}(\varphi) \in \mathcal{S}_G(\mathfrak{A})$ in terms of the one associated to φ itself.

Proposition II.4.1
Let $(\mathfrak{A}, G, \alpha)$ be a C^*-system and $\varphi \in \mathcal{S}(\mathfrak{A})$. Then,

(1) $(\mathcal{H}_{\varphi|_{\mathfrak{A}^G}}, \pi_{\varphi|_{\mathfrak{A}^G}}, \xi_{\varphi|_{\mathfrak{A}^G}}) = (\mathcal{K}_\varphi, r_{\mathcal{K}_\varphi} \circ \pi_\varphi|_{\mathfrak{A}^G}, \xi_\varphi)$, where $\mathcal{K}_\varphi := \overline{\pi_\varphi(\mathfrak{A}^G)\xi_\varphi} \subseteq \mathcal{H}_\varphi$

(2) $(\mathcal{H}_{E_G^{\mathrm{t}}\varphi}, \pi_{E_G^{\mathrm{t}}\varphi}, \xi_{E_G^{\mathrm{t}}\varphi}) = (\mathcal{L}, r_{\mathcal{L}} \circ \Pi, V\xi_\varphi)$ where $(\mathcal{H}, \Pi, V)$ is the *minimal Stinespring dilation* of the c.p.u. map $r_{\mathcal{K}_\varphi} \circ \pi_\varphi \circ E_G \colon \mathfrak{A} \to \mathcal{B}(\mathcal{K}_\varphi)$ and $\mathcal{L} := \overline{\Pi(\mathfrak{A})V\xi_\varphi} \subseteq \mathcal{H}$, i.e.

$$\begin{cases} \Pi \colon \mathfrak{A} \to \mathcal{B}(\mathcal{H}) \text{ unital representation} \\ V \in \mathcal{B}(\mathcal{K}_\varphi, \mathcal{H}) \text{ isometry} \\ \pi_\varphi(E_G(\cdot))|_{\mathcal{K}_\varphi} = P_{\mathcal{K}_\varphi}\Pi(\cdot)|_{\mathcal{K}_\varphi} \\ \overline{\Pi(\mathfrak{A})V(\mathcal{K}_\varphi)} = \mathcal{H} \end{cases} \qquad \begin{array}{ccc} \mathcal{K}_\varphi & \xrightarrow{(\pi_\varphi \circ E_G)(a)} & \mathcal{K}_\varphi \\ {\scriptstyle V}\big\downarrow & \circlearrowleft & \big\uparrow{\scriptstyle P_{\mathcal{K}_\varphi}} \\ \mathcal{H} & \xrightarrow[\Pi(a)]{} & \mathcal{H} \end{array} \qquad (a \in \mathfrak{A})$$

Moreover, there exists a (strongly continuous) unitary representation $U \colon G \to \mathcal{U}(\mathcal{L})$ s.t.

$$\begin{cases} U_g V\xi_\varphi = V\xi_\varphi, \quad g \in G \\ (\Pi \circ \alpha_g)(\cdot)|_{\mathcal{L}} = U_g \Pi(\cdot) U_g^*, \quad g \in G \end{cases} \qquad \begin{array}{ccc} \mathcal{L} & \xrightarrow{(\Pi \circ \alpha_g)(a)} & \mathcal{L} \\ {\scriptstyle U_g^*}\big\downarrow & \circlearrowleft & \big\uparrow{\scriptstyle U_g} \\ \mathcal{L} & \xrightarrow[\Pi(a)]{} & \mathcal{L} \end{array} \qquad (a \in \mathfrak{A}, g \in G)$$

As a special case, if $\varphi \in \mathcal{S}_G(\mathfrak{A})$, then

(i) $\mathcal{K}_\varphi = \mathcal{H}_\varphi^G$

(ii) $(\mathcal{H}, \Pi, V) = (\mathcal{H}_\varphi, \pi_\varphi, \iota_{\mathcal{H}_\varphi^G})$, with $V^* = P_\varphi \colon \mathcal{H}_\varphi \twoheadrightarrow \mathcal{H}_\varphi^G$

(iii) $\mathcal{L} = \mathcal{H}_\varphi$

Proof.
(1) The proof is straightforward since, on $\mathfrak{A}^G$, $\varphi(\cdot) = \langle (r_{\mathcal{K}_\varphi} \circ \pi_\varphi)(\cdot)\xi_\varphi, \xi_\varphi\rangle$ and $r_{\mathcal{K}_\varphi} \circ \pi_\varphi|_{\mathfrak{A}^G} \colon \mathfrak{A}^G \to \mathcal{B}(\mathcal{K}_\varphi)$ is a cyclic representation of $\mathfrak{A}^G$ with cyclic vector $\xi_\varphi \in \mathcal{K}_\varphi$.
(2) Again, the proof is easy since, on $\mathfrak{A}$, $(E_G^{\mathrm{t}}\varphi)(\cdot) = \langle \pi_\varphi(E_G(\cdot))\xi_\varphi, \xi_\varphi\rangle = \langle \Pi(\cdot)V\xi_\varphi, V\xi_\varphi\rangle$ and $r_{\mathcal{L}} \circ \Pi \colon \mathfrak{A} \to \mathcal{B}(\mathcal{L})$ is a cyclic representation of $\mathfrak{A}$ with cyclic vector $V\xi_\varphi \in \mathcal{L}$. Moreover, $E_G^{\mathrm{t}}\varphi \in \mathcal{S}_G(\mathfrak{A})$ whence there exists a unitary representation U of G on $\mathcal{L}$ satisfying the covariance property.
Now, suppose that $\varphi \in \mathcal{S}_G(\mathfrak{A})$.
(i) $E_G \colon \mathfrak{A} \twoheadrightarrow \mathfrak{A}^G$ passes to the quotient modulo the left kernel $\mathfrak{n}_\varphi$ of φ, yielding a contractive, idempotent (hence, positive) surjective map of pre-Hilbert spaces $\widetilde{E_G} \colon \mathfrak{A}/\mathfrak{n}_\varphi \twoheadrightarrow \mathfrak{A}^G/(\mathfrak{n}_\varphi \cap \mathfrak{A}^G)$. Observe that for every $a \in \mathfrak{A}$

$$\widetilde{E_G}[a]_\varphi = \left[\int_G \alpha_g(a)\mathrm{d}g\right]_\varphi = \int_G U_\varphi(g)[a]_\varphi \mathrm{d}g = P_\varphi[a]_\varphi$$

thus the unique bounded extension of $\widetilde{E_G}$ to $\mathcal{H}_\varphi$ coincides with the orthogonal projection $P_\varphi\colon \mathcal{H}_\varphi \twoheadrightarrow \mathcal{H}_\varphi^G$. Furthermore, since $\xi_\varphi \in \mathcal{H}_\varphi^G$, $\pi_\varphi(\mathfrak{A}^G)\xi_\varphi \subseteq \mathcal{H}_\varphi^G$ whence $\mathcal{K}_\varphi = \overline{\pi_\varphi(\mathfrak{A}^G)\xi_\varphi} \subseteq \mathcal{H}_\varphi^G$. For the converse inclusion, let $\xi \in \mathcal{H}_\varphi^G$ i.e. $P_\varphi \xi = \int_G U_\varphi(g)\xi \mathrm{d}g = \xi$. By cyclicity of the vector $\xi_\varphi \in \mathcal{H}_\varphi$ for π_φ, there exists a sequence $(a_n)_{n\in\mathbb{N}} \subset \mathfrak{A}$ s.t. $\lim_{n\to+\infty} \pi_\varphi(a_n)\xi_\varphi = \xi$. In particular $\{\pi_\varphi(a_n)\xi_\varphi\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}_\varphi$ by a constant $M > 0$, therefore the sequence $\varphi_n\colon g \mapsto U_\varphi(g)\pi_\varphi(a_n)\xi_\varphi$ is uniformly bounded:

$$\|\varphi_n(g)\|_{\mathcal{H}_\varphi} = \|\pi_\varphi(a_n)\xi_\varphi\|_{\mathcal{H}_\varphi} \leq M\,.$$

By Lebesgue Dominated Convergence Theorem, covariance and continuity of π_φ,

$$\xi = \lim_{n\to+\infty} \int_G U_\varphi(g)\pi_\varphi(a_n)\xi_\varphi \mathrm{d}g = \lim_{n\to+\infty} \int_G \pi_\varphi(\alpha_g(a_n))\xi_\varphi \mathrm{d}g = \lim_{n\to+\infty} \left(\int_G \pi_\varphi(\alpha_g(a_n))\mathrm{d}g \right) \xi_\varphi =$$

$$= \lim_{n\to+\infty} \pi_\varphi \left(\int_G \alpha_g(a_n)\mathrm{d}g \right) \xi_\varphi = \lim_{n\to+\infty} \pi_\varphi(E_G(a_n))\xi_\varphi$$

whence $\xi \in \overline{\pi_\varphi(\mathfrak{A}^G)\xi_\varphi}$, and $\mathcal{K}_\varphi = \mathcal{H}_\varphi^G$.
(ii) $(\mathcal{H}_\varphi, \pi_\varphi, \iota_{\mathcal{H}_\varphi^G})$ is a Stinespring dilation for the c.p.u. map $r_{\mathcal{H}_\varphi^G} \circ \pi_\varphi \circ E_G$. Since $\overline{\pi_\varphi(\mathfrak{A})\mathcal{H}_\varphi^G} = \mathcal{H}_\varphi$, it is also the minimal one, up to unitary equivalence. In particular, $V^* = \iota^*_{\mathcal{H}_\varphi^G} = P_\varphi$.
(iii) By (ii), we immediately have $\mathcal{L} = \mathcal{H}_\varphi$. □

Remark II.4.2
If $\varphi \in \mathcal{S}_G(\mathfrak{A})$, then

- $\pi_\varphi(E_G(\cdot)) = P_\varphi \pi_\varphi(\cdot)$ as operators acting on $\mathcal{H}_\varphi^G$
- the GNS representations of φ and $\varphi|_{\mathfrak{A}^G}$ are related by the inequality

$$\|\pi_{\varphi|_{\mathfrak{A}^G}}(E_G(\cdot))\|_{\mathcal{B}(\mathcal{H}_\varphi^G)} \leq \|\pi_\varphi(\cdot)\|_{\mathcal{B}(\mathcal{H}_\varphi)}\,.$$

We recall now a construction which will be useful in the sequel (for a reference, see p. 139 in [86]). Any representation $(\pi, \mathcal{H}_\pi)$ of $\mathfrak{A}$ induces a covariant representation $(\pi^G, \lambda_G^\pi) \in \mathsf{Cov}(\mathfrak{A}, G, \alpha)$ as follows. Let $\mathcal{H}_{\pi^G} := L^2(G, \mathrm{d}g; \mathcal{H}_\pi) \cong \mathcal{H}_\pi \otimes L^2(G, \mathrm{d}g)$ and for every $\xi \in \mathcal{H}_{\pi^G}$ and $a \in \mathfrak{A}$ define

$$\begin{aligned} \pi^G(a)(\xi)\colon G &\to \mathcal{H}_{\pi^G} \\ g &\mapsto \pi(\alpha_{g^{-1}}(a))(\xi_g) \end{aligned} \tag{II.3}$$

It is easy to see that, if $\lambda_G^\pi := I_{\mathcal{H}_\pi} \otimes \lambda_G$ is the *ampliation* of the left regular representation λ_G on $\mathcal{H}_{\pi^G}$, then

$$\lambda_G^\pi(g)\pi^G(a)\lambda_G^\pi(g^{-1}) = \pi^G(\alpha_g(a)) \quad a \in \mathfrak{A},\, g \in G\,.$$

Therefore, $(\pi^G, \mathcal{H}_{\pi^G}, \lambda_G^\pi)$ becomes a covariant representation. Notice that, if π is faithful, then $\overline{\langle \pi^G(\mathfrak{A}), \lambda_G^\pi(G)\rangle}^{\mathcal{B}(\mathcal{H}_{\pi^G})}$ does not depend on the chosen *faithful* representation π of $\mathfrak{A}$ and provides the definition of the *reduced crossed product* $\mathfrak{A} \rtimes_{\alpha,\mathrm{r}} G$ (see Definition 4.1.4 and Proposition 4.1.5 in p. 118: there, Brown and Ozawa treat the discrete case only, but the proof can be adapted to the compact one as well). The *folium* of π^G is defined as

$$\mathcal{F}(\pi^G) := \left\{ \eta\colon a \mapsto \sum_{u\in N} \int_G \langle u(g), (A\pi^G(a)u)(g)\rangle_{\mathcal{H}_\pi} \mathrm{d}g \right\}_{\substack{A\in\mathcal{B}_1(\mathcal{H}_{\pi^G}) \\ \text{positive,unit trace}}}$$

where N is any orthonormal basis on $\mathcal{H}_{\pi^G}$. If π^G is faithful, $\overline{\mathcal{F}(\pi^G)}^{\mathrm{w}^*} = \mathfrak{A}^*$ (topological dual of $\mathfrak{A}$) and $\overline{\mathcal{F}(\pi^G)_+}^{\mathrm{w}^*} = \mathfrak{A}^*_+$ (positive cone in the topological dual $\mathfrak{A}^*$).

Let $\widehat{G}$ be the family of (unitary equivalence classes of) strongly continuous, unitary representations of G on some Hilbert space, which are *irreducible*: for each of them, the only proper, stable, closed subspace is the trivial one. By the Schur lemma, we can write

$$\widehat{G} = \{\sigma \in \mathsf{URep}(G) \colon \sigma(G)' = \mathbb{C} I_{\mathcal{H}_\sigma}\} = \{\sigma \in \mathsf{URep}(G) \colon \sigma(G)'' = \mathcal{B}(\mathcal{H}_\sigma)\} = \\ = \{\sigma \in \mathsf{URep}(G) \colon \text{every } \xi \in \mathcal{H}_\sigma \setminus \{0\} \text{ cyclic for } \sigma\}\,.$$

For every $\sigma \in \mathsf{URep}(G)$, let $d_\sigma := \dim_{\mathbb{C}} \mathcal{H}_\sigma \in \mathbb{N} \cup \{\infty\}$. This is a good definition, being independent on the particular choice of representative of the unitary equivalence class. Also, for every $\sigma \in \mathsf{URep}(G)$ and $\xi, \eta \in \mathcal{H}_\sigma$, let $\omega_{\sigma,\xi,\eta} := \langle \xi, \cdot\, \eta\rangle \in \mathcal{B}(\mathcal{H}_\sigma)_*$. Then, the classical Peter-Weyl theorem guarantees that

- $\widehat{G} \subset \{\sigma \in \mathsf{URep}(G) \colon d_\sigma < \infty\}$
- if $\sigma \in \mathsf{URep}(G)$, then σ is the direct sum of elements in $\widehat{G}$. In particular, for each pair $\sigma, \tau \in \widehat{G}$, there exists a finite set $F_{\sigma,\tau} \subset \widehat{G}$ s.t. $\sigma \otimes \tau = \bigoplus_{\rho \in F_{\sigma,\tau}} \Gamma^\rho_{\sigma,\tau}\, \rho$ for some structure constants $\Gamma^\rho_{\sigma,\tau} \neq 0$, $\rho \in F_{\sigma,\tau}$. Observe that they must satisfy $\sum_{\rho \in F_{\sigma,\tau}} \Gamma^\rho_{\sigma,\tau} d_\rho = d_\sigma d_\tau$.
- the left regular representation $\lambda_G \colon G \to \mathcal{U}(L^2(G, \mathrm{d}g))$ is unitarily equivalent to $\bigoplus_{\sigma \in \widehat{G}} d_\sigma \sigma$
- $\{\omega_{\sigma,\xi,\eta} \circ \sigma \colon \sigma \in \widehat{G}, \xi, \eta \in \mathcal{H}_\sigma\}$ separates the points of G, and hence by Stone-Weierstrass theorem
$$\mathcal{C}(G) = [\omega_{\sigma,\xi,\eta} \circ \sigma \colon \sigma \in \widehat{G}, \xi, \eta \in \mathcal{H}_\sigma]\,.$$
For $\sigma \in \widehat{G}$, by identifying $\mathcal{H}_\sigma$ with $\mathbb{C}^{d_\sigma}$ and defining $\sigma(\cdot)_{ij} := \omega_{\sigma,e_i,e_j} \circ \sigma$, with $\{e_i\}_{i=1}^{d_\sigma} \subseteq \mathbb{C}^{d_\sigma}$ an orthonormal basis, $\mathcal{C}(G) = [\sigma(\cdot)_{ij} \colon \sigma \in \widehat{G}, i, j = 1, \dots, d_\sigma]$

For $\sigma \in \widehat{G}$, let $\chi_\sigma(\cdot) := \dfrac{\mathrm{tr}(\sigma(\cdot))}{d_\sigma} = \dfrac{1}{d_\sigma} \sum_{i=1}^{d_\sigma} \sigma(\cdot)_{ii}$ be the *(normalized) character* of σ (once more, the definition does not depend on the choice of the class representative). For each $\sigma \in \widehat{G}$, we thus define the Bochner integrals

$$E_\sigma(a) := \int_G \overline{\chi_\sigma(g)} \alpha_g(a) \mathrm{d}g, \quad a \in \mathfrak{A}\,. \tag{II.4}$$

$$E_{\sigma,ij}(a) := \int_G \overline{\sigma(g)_{ji}} \alpha_g(a) \mathrm{d}g, \quad a \in \mathfrak{A},\ i, j = 1, \dots, d_\sigma\,. \tag{II.5}$$

Then, E_σ is a contractive projection from $\mathfrak{A}$ onto the closed operator space $\mathfrak{A}_\sigma := \{a \in \mathfrak{A} \colon E_\sigma(a) = a\}$. It is worth noticing that E_σ (and, consequently, $\mathfrak{A}_\sigma$) does *not* depend on the choice of the class representative, whereas $E_{\sigma,ij}$ $(i, j = 1, \dots, d_\sigma)$ does. Nonetheless, it always results that $E_{\sigma,ij}(\mathfrak{A}) \subset \mathfrak{A}_\sigma$ for every $i, j = 1, \dots, d_\rho$ (see Section 2 in [63], p. 278-279). In particular, if $\iota = e_{\widehat{G}}$ is the trivial representation, $E_\iota = E_G$ and $\mathfrak{A}_\iota = \mathfrak{A}^G$, so that if $\omega \in \mathcal{S}(\mathfrak{A})$ is the unique G-invariant state of an ergodic C^*-system $(\mathfrak{A}, G, \alpha)$ we can write $E_\iota(a) = \omega(a) \mathbb{1}_{\mathfrak{A}}$ $(a \in \mathfrak{A})$. Furthermore, $\mathfrak{A}_o := \dot{+}_{\sigma \in \widehat{G}} \mathfrak{A}_\sigma$ is uniformly dense in $\mathfrak{A}$ (see Lemma 2.4 in [63], p. 279).

We will call $\mathfrak{A}_o$ the *algebraic layer* of the C^*-system $(\mathfrak{A}, G, \alpha)$, a G-stable pre-C^*-subalgebra of $\mathfrak{A}$.

When the compact group G is abelian, $\widehat{G} = \mathsf{Hom}(G, \mathbb{T})$ is a discrete abelian group, and $\mathfrak{A}_o$ admits a very well-behaved structure, as the following proposition will show. Firstly, let $\mathfrak{A}$ be a unital C^*-algebra and G a compact abelian group acting by translation on the (possibly, non-abelian) C^*-algebra $\mathcal{C}(G, \mathfrak{A})$ via $G \overset{\alpha}{\curvearrowright} \mathcal{C}(G, \mathfrak{A})$. Then, we have an isometric *-isomorphism of pre-C^*-algebras, given by the abstract Fourier transform:

$$\begin{aligned} \mathcal{F}\colon (\mathcal{C}_c(\widehat{G}, \mathfrak{A}, \tau), \|\cdot\|_{\mathrm{f}}) &\to (\mathcal{C}(G, \mathfrak{A})_o, \|\cdot\|_\infty) \\ a \otimes u_\sigma &\mapsto [g \mapsto \sigma(g)a] \end{aligned}$$

where $\mathcal{C}_c(\widehat{G}, \mathfrak{A}, \tau)$ is the convolution algebra of the C^*-system $(\widehat{G}, \mathfrak{A}, \tau)$ where $\widehat{G}$ acts trivially on $\mathfrak{A}$ via $\widehat{G} \overset{\tau}{\curvearrowright} \mathfrak{A}$: it is a pre-$C^*$-algebra w.r.t. the *full crossed product norm* $\|\cdot\|_{\mathrm{f}}$ and its completion gives $\mathfrak{A} \rtimes_{\tau,\mathrm{f}} \widehat{G} \cong \mathfrak{A} \otimes_{\min} C^*(\widehat{G})$. Precisely, for every $\sigma \in \widehat{G}$,

$$\mathcal{F}(\{a \otimes u_\sigma\}_{a\in\mathfrak{A}}) = \{g \mapsto \sigma(g)a\}_{a\in\mathfrak{A}} = \{f \in \mathcal{C}(G, \mathfrak{A})\colon \alpha_g(f) = \sigma(g)f\} = \mathcal{C}(G, \mathfrak{A})_\sigma$$

and $\mathcal{F}$ extends to an isometric *-isomorphism of C^*-algebras: $\mathfrak{A} \otimes_{\min} C^*(\widehat{G}) \cong \mathcal{C}(G, \mathfrak{A})$. Notice also that $\{g \mapsto \sigma(g)a\}_{a\in\mathfrak{A}} = \mathfrak{A}[\sigma]$, the cyclic $\mathfrak{A}$-module generated by σ.

Proposition II.4.3
Let G be a compact abelian group and $\mathfrak{A}$ a unital C^*-algebra. The following data are equivalent:

(i) an action $G \overset{\alpha}{\curvearrowright} \mathfrak{A}$: $\alpha \in \mathsf{Hom}(G, \mathsf{Aut}(\mathfrak{A}))$ s.t. $g \mapsto \alpha_g(a)$ is norm-continuous for each $a \in \mathfrak{A}$

(ii) a coaction $C^*(\widehat{G}) \overset{\delta}{\curvearrowright} \mathfrak{A}$: $\delta \in \mathsf{Hom}(\mathfrak{A}, \mathcal{C}(G, \mathfrak{A}))$ injective s.t.

- $[\delta(\mathfrak{A})] = \mathcal{C}(G, \mathfrak{A})$ (non-degeneracy)
- the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{\ \delta\ } & \mathcal{C}(G, \mathfrak{A}) \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\delta\otimes\mathrm{id}_G} \\ \mathcal{C}(G, \mathfrak{A}) & \xrightarrow[\mathrm{id}_{\mathfrak{A}}\otimes\delta_G]{} & \mathcal{C}(G\times G, \mathfrak{A}) \end{array} \qquad \begin{aligned} \delta_G\colon \mathcal{C}(G) &\to \mathcal{C}(G\times G) \\ f &\mapsto [(\sigma,\tau) \mapsto f(\sigma\tau)] \end{aligned}$$

(iii) a topological $\widehat{G}$-grading on $\mathfrak{A}$: there exists a family $\{\mathfrak{A}_\sigma\}_{\sigma\in\widehat{G}}$ of linearly independent, Banach subspaces of $\mathfrak{A}$ such that

(iii-a) $\mathfrak{A} = \bigoplus_{\sigma\in\widehat{G}} \mathfrak{A}_\sigma$

(iii-b) $\mathfrak{A}_\sigma\mathfrak{A}_\tau \subseteq \mathfrak{A}_{\sigma\tau}, \quad \sigma,\tau \in \widehat{G}$

(iii-c) $\mathfrak{A}_\sigma^* = \mathfrak{A}_{\overline{\sigma}}, \quad \sigma \in \widehat{G}$

(iii-d) there exists $E \in \mathcal{B}(\mathfrak{A})$ s.t. $\begin{cases} E(a) = a, \ a \in \mathfrak{A}_\iota \\ E(a) = 0, \ a \in \mathfrak{A}_\sigma, \sigma \neq \iota \end{cases}$

Proof.
The equivalence (i)⇔(ii) is exposed in Example A.23 of [89] (p.127). For the convenience of the

reader, let us sketch the proof.

(i)⇒(ii): Suppose that G acts on $\mathfrak{A}$ via α. Then,

$$\begin{aligned} \delta^\alpha \colon \mathfrak{A} &\hookrightarrow \mathcal{C}(G, \mathfrak{A}) \\ a &\mapsto [g \mapsto \alpha_g(a)] \end{aligned}$$

is a non-degenerate (i.e. $\delta^\alpha(\mathfrak{A})$ is dense in $\mathcal{C}(G, \mathfrak{A})$) *-homomorphism making the diagram

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{\delta^\alpha} & \mathcal{C}(G, \mathfrak{A}) \\ {\scriptstyle \delta^\alpha}\downarrow & & \downarrow{\scriptstyle \delta^\alpha \otimes \mathrm{id}_G} \\ \mathcal{C}(G, \mathfrak{A}) & \xrightarrow[\mathrm{id}_{\mathfrak{A}} \otimes \delta_G]{} & \mathcal{C}(G \times G, \mathfrak{A}) \end{array}$$

commute: for every $a \in \mathfrak{A}$, $((\delta^\alpha \otimes \mathrm{id}_G) \circ \delta^\alpha)(a) = ((\mathrm{id}_{\mathfrak{A}} \otimes \delta_G) \circ \delta^\alpha)(a) : (g, h) \mapsto \alpha_{gh}(a)$.

(ii)⇒(i): Suppose that $\widehat{G}$ coacts on $\mathfrak{A}$ via δ. Then,

$$\begin{aligned} \alpha^\delta \colon G &\to \mathsf{Aut}(\mathfrak{A}) \\ g &\mapsto [a \mapsto \delta(a)(g)] \end{aligned}$$

defines a strongly continuous action of G on $\mathfrak{A}$.

(i)⇒(iii): Suppose that G acts on $\mathfrak{A}$ via α. For every $\sigma \in \widehat{G}$, define the *σ-spectral subspace*

$$\mathfrak{A}_\sigma := \{a \in \mathfrak{A} \colon \alpha_g(a) = \sigma(g)a,\ g \in G\}$$

Then, one straightforwardly checks that $\{\mathfrak{A}_\sigma\}_{\sigma \in \widehat{G}}$ is a family of linearly independent, norm-closed subspaces of $\mathfrak{A}$. Moreover,

$$\begin{cases} \mathfrak{A}_\sigma \mathfrak{A}_\tau \subseteq \mathfrak{A}_{\sigma\tau}, & \sigma, \tau \in \widehat{G} \\ \mathfrak{A}_\sigma^* = \mathfrak{A}_{\overline{\sigma}}, & \sigma \in \widehat{G} \\ \mathfrak{A}_o := \dot{+}_{\sigma \in \widehat{G}} \mathfrak{A}_\sigma \text{ dense in } \mathfrak{A} \end{cases}$$

Lastly, $E := E_G \in \mathcal{B}(\mathfrak{A})$ in Equation II.2 satisfies $E(a) = a$ for $a \in \mathfrak{A}_\iota = \mathfrak{A}^G$ and $E(a) = 0$ for $a \in \mathfrak{A}_\sigma$, $\sigma \neq \iota$.

(iii)⇒(i) is established in Theorem 3 of [68] (p. 129). In particular, there is a strongly continuous action $G \overset{\alpha}{\curvearrowright} \mathfrak{A}$ such that $\alpha_g(a) = \sigma(g)a$ for every $a \in \mathfrak{A}_\sigma$, $\sigma \in \widehat{G}$, and $E = \int_G \alpha_g(\cdot)\mathrm{d}g = E_G \in \mathcal{B}(\mathfrak{A})$. □

Remark II.4.4
The points (iii-a), (iii-b) and (iii-c) in Proposition II.4.3 gives the definition of *$\widehat{G}$-grading*. Its *topological* feature is encoded in point (iii-d), and a map $E \in \mathcal{B}(\mathfrak{A})$ satisfying (iii-d) is the unique one that does so, and it is automatically an expectation of $\mathfrak{A}$ onto $\mathfrak{A}_\iota$ (see Theorem 3.3 in [26], p. 50). We also remark that the definition of $\mathfrak{A}_\sigma$ in the proof of Proposition II.4.3 matches the one previously given in the general (possibly, non-abelian) setting. In particular, we still have a family of contractive projections $E_\sigma = \int_G \sigma(g)\alpha_g(\cdot)\mathrm{d}g$ for every $\sigma \in \widehat{G}$.

Remark II.4.5
Suppose that G is *monothetic*, namely it admits a dense cyclic subgroup. Let $\mathcal{G}_G := \{g \in G \colon \langle g \rangle \leq G \text{ dense}\}$ the family of generators of G. Then, it is easy to show that if $g \in \mathcal{G}_G$,

- $\alpha_g^{n_j} \xrightarrow{j\uparrow+\infty} \mathrm{id}_{\mathfrak{A}}$ pointwise in norm, for a suitable subsequence $\{n_j\}_{j\in\mathbb{N}} \subset \mathbb{N}$
- $\mathcal{S}_{\alpha_g}(\mathfrak{A}) = \mathcal{S}_G(\mathfrak{A})$

Examples of compact monothetic groups are, of course, cyclic ones, as well as $\mathbb{T}^n$ $(n \geq 1)$ and the Bohr compactification of $\mathbb{Z}$. More generally, G is monothetic iff $\widehat{G}$ is continuously embedded into the discrete torus $\mathbb{T}_{\mathrm{d}}$ (see Theorem I in [45], p. 255).

Remark II.4.6
Equivalence of (i) and (ii) in Proposition II.4.3 has been established in the far more general setting of (possibly, non-abelian) locally compact groups, for which there is a 1-1 correspondence between G-actions and $\widehat{C^*_{\mathrm{r}}(G)}$-coactions, with $\widehat{C^*_{\mathrm{r}}(G)}$ being the dual C^*-quantum group of the reduced group algebra of G (see [5], where Baaj-Skandalis duality yields $\widehat{C^*_{\mathrm{r}}(G)} \cong \mathcal{C}_0(G)$). It is also worth noticing that every $\widehat{G}$-grading on $\mathfrak{A}$ gives rise to a *Fell bundle over* $\widehat{G}$. A Fell (or C^*-algebraic) bundle over any topological group $\mathcal{G}$ is a quadruplet $\mathscr{B} := \langle X, \pi, \cdot, ^* \rangle$ s.t.

- X Hausdorff space (*bundle space*)
- $\pi\colon X \twoheadrightarrow \mathcal{G}$ τ_X-continuous and open (*bundle projection*, with *fibers* $X_\sigma := \pi^{-1}(\sigma)$ and *sections* $f\colon \mathcal{G} \to X$ i.e. $f(\sigma) \in X_\sigma$, $\sigma \in \mathcal{G}$)
- $(X_\sigma, \|\cdot\|_\sigma)$ Banach space, $\sigma \in \mathcal{G}$
- $X_\sigma \ni a \mapsto \|a\|_\sigma \in [0, +\infty)$ τ_X-continuous, $\sigma \in \mathcal{G}$
- $+_\sigma\colon X_\sigma \to X_\sigma$ τ_X-continuous, $\sigma \in \mathcal{G}$ (continuity of addition)
- $\lambda_\sigma\colon X_\sigma \to X_\sigma$ τ_X-continuous, $\sigma \in \mathcal{G}$, $\lambda \in \mathbb{C}$ (continuity of product-by-scalar)
- $\begin{cases} \|a_\lambda\|_{\sigma_\lambda} \xrightarrow{\lambda} 0 \text{ in } \mathbb{R} \\ \pi(a_\lambda) \xrightarrow{\lambda} \sigma \text{ in } \tau_{\mathcal{G}} \end{cases}$ implies $a_\lambda \xrightarrow{\lambda} 0_\sigma$ in τ_X
- $\cdot\colon X \times X \to X$ associative, τ_X-continuous and s.t. $\pi(a \cdot b) = \pi(a)\pi(b)$ $(a, b \in X)$; $\cdot|_{X_\sigma \times X_\tau}\colon X_\sigma \times X_\tau \to X_{\sigma\tau}$ bilinear, $\|a \cdot b\|_{\sigma\tau} \leq \|a\|_\sigma \|b\|_\tau$ $(a \in X_\sigma, b \in X_\tau)$
- $^*\colon X \to X$ involutive, isometric, τ_X-continuous, $\cdot$-reversing and s.t. $\pi(a^*) = \overline{\pi(a)}$ $(a \in X)$; $^*|_{X_\sigma}\colon X_\sigma \to X_{\sigma^{-1}}$ antilinear
- $\|a^* a\| = \|a\|^2$ for every $a \in X$
- $a^* a \geq 0$ in X_0 for every $a \in X$

In particular, X_0 is a C^*-algebra (for a reference, see Definitions 13.4 in [91] at p. 127 and 16.2 at p. 871 in [92]). In our case, $\mathcal{G} = \widehat{G}$ and $X = \coprod_{\sigma \in \widehat{G}} \mathfrak{A}_\sigma$. Moreover, the algebraic layer $\mathfrak{A}_o$ can be viewed as the *algebra of finitely supported sections*

$$\mathcal{C}_c(\mathscr{B}) := \left\{ f\colon \widehat{G} \to \coprod_{\sigma \in \widehat{G}} \mathfrak{A}_\sigma \colon f \text{ section}, \ |\{f(\sigma) \neq 0\}| < \infty \right\},$$

an involutive algebra if endowed with the convolution product

$$(f * g)(\sigma) := \sum_{\tau \in \widehat{G}} f(\tau) g(\tau^{-1}\sigma), \quad \sigma \in \widehat{G}, \quad f, g \in \mathcal{C}_c(\mathscr{B})$$

and the involution $(f^*)(\sigma) := f(\sigma^{-1})^*$, $\sigma \in \widehat{G}$, $f \in \mathcal{C}_c(\mathscr{B})$.

Remark II.4.7
Consider the subset of $\widehat{G}$ $\mathrm{spec}_o(\alpha) := \left\{\sigma \in \widehat{G} \colon \mathfrak{A}_\sigma \neq \{0\}\right\}$. Then, $\mathfrak{A}_o = \mathfrak{A}$ if and only if $\mathrm{spec}_o(\alpha)$ is finite. Moreover, $\mathrm{spec}_o(\alpha)$ contains ι and is symmetric in $\widehat{G}$ (stable under inverse operation), though it is not true in general that it is a subgroup of $\widehat{G}$. Thus, $\mathrm{spec}(\alpha)$ is defined as the subgroup of $\widehat{G}$ generated by $\mathrm{spec}_o(\alpha)$. When α is ergodic, taking *verbatim* the proof of Proposition 2.2 of [28] (p. 312), $\mathfrak{A}_\sigma$ with $\sigma \in \mathrm{spec}_o(\alpha)$ is always a one-dimensional subspace made of all scalar multiples of a single unitary. As a consequence, in such a case $\mathrm{spec}_o(\alpha) = \mathrm{spec}(\alpha)$ is a subgroup of $\widehat{G}$. For the convenience of the reader, we report the proof here below.

Proof.
Let $\sigma \in \mathrm{spec}_o(\alpha)$. Consider two non-zero $a, b \in \mathfrak{A}_\sigma$. Since $a^*a, b^*b \in \mathfrak{A}^G = \mathbb{C}\mathbb{1}$ and they both are positive, we have $a^*a = aa^* = \|a\|^2\mathbb{1}$ and $b^*b = bb^* = \|b\|^2\mathbb{1}$. Up to normalization, we can thus suppose a, b unitary. It follows that $a^*b \in \mathfrak{A}^G = \mathbb{C}\mathbb{1}$ is unitary, so that $a^*b = e^{i\alpha(a^*b)}\mathbb{1}$ i.e. $b = e^{i\alpha(a^*b)}a$. Hence, $\mathfrak{A}_\sigma = \mathbb{C}a$. All things considered, if $\sigma, \tau \in \mathrm{spec}_o(\alpha)$, there exists $u_\sigma, u_\tau \in \mathcal{U}(\mathfrak{A})$ s.t. $\mathfrak{A}_\sigma = \mathbb{C}u_\sigma$ and $\mathfrak{A}_\tau = \mathbb{C}u_\tau$. In particular, $\mathfrak{A}_{\sigma^{-1}} = \mathbb{C}u_\sigma^*$ and $\mathfrak{A}_{\sigma\tau} = \mathbb{C}u_\sigma u_\tau$. We conclude that $\sigma^{-1}, \sigma\tau \in \mathrm{spec}_o(\alpha)$ and $\mathrm{spec}(\alpha) = \mathrm{spec}_o(\alpha)$. □

We conclude this section by observing that, exploiting the pre-Hilbert *-bimodule structure of $(\mathfrak{A}, \langle\cdot,\cdot\rangle_{E_G})$ over its fixed point subalgebra $\mathfrak{A}^G$ where $\langle a_1, a_2\rangle_{E_G} = \int_G \alpha_g(a_1)\alpha_g(a_2)\mathrm{d}g$, $a_1, a_2 \in \mathfrak{A}$, we easily see that the spectral subspaces $\{\mathfrak{A}_\sigma\}_{\sigma\in\widehat{G}}$ are pairwise orthogonal w.r.t. both the sesquilinear $\mathfrak{A}^G$-valued maps $\langle\cdot,\cdot\rangle_l$ and $\langle\cdot,\cdot\rangle_r$. The direct sum decomposition of the algebraic layer $\mathfrak{A}_o$ can then be thought of as an *orthogonal decomposition*, in both the left and right pre-Hilbert $\mathfrak{A}^G$-module structure of $\mathfrak{A}$.

II.5. On the automatic continuity of representations

We now establish a result which will play a crucial role in the sequel. It concerns the automatic boundedness of *all* representations of the algebraic layer $\mathfrak{A}_o$ of a C^*-system $(\mathfrak{A}, G, \alpha)$, with G compact and abelian.
We start by showing that every bounded *-algebra A admits an *enveloping C^*-algebra* $C^*(A)$ (for a reference, see Proposition 1.3 in [62], p. 2707).

Lemma II.5.1
Let A be a bounded *-algebra. Then,

$$\sup_{\mathsf{Rep}(A)} \|\pi(a)\| < \infty, \quad a \in A.$$

In particular, $\|\cdot\|_{\mathrm{u},o} := \sup_{\mathsf{Rep}(A)} \|\pi(\cdot)\|$ is a well-defined C^*-seminorm on A and $\mathcal{N} := \{a \in A \colon \|a\|_{\mathrm{u},o} = 0\}$ is a two-sided *-ideal of A, thus yielding a C^*-norm on $A/\mathcal{N}$ defined by $\|a + \mathcal{N}\|_{\mathrm{u}} := \|a\|_{\mathrm{u},o}$, $a \in A$.

Proof.
By contradiction, if there exists $a \in A$ such that $\sup_{\mathsf{Rep}(A)} \|\pi(a)\| = +\infty$, there must exist a net $\{\pi_i\}_{i\in I} \subset \mathsf{Rep}(A)$ s.t. $\lim_i \|\pi_i(a)\| = +\infty$. Consider the representation $\pi := \dotplus_{i\in I} \pi_i$ on the algebraic direct sum $\mathcal{H}_o := \dotplus_{i\in I} \mathcal{H}_i$. By construction, $\pi(a) \in \mathcal{L}(\mathcal{H}_o)$ is an unbounded operator: a contradiction, since A is a bounded *-algebra. The rest of the assertion is routinary. □

If A is a bounded *-algebra, let $C^*(A) := \overline{A/\mathcal{N}}^{\,\mathrm{u}}$ be the enveloping C^*-algebra of A.
From now on, we will deal with C^*-systems $(\mathfrak{A}, G, \alpha)$ where G is compact and abelian. In this setting, $\varphi \in \mathcal{S}_G(\mathfrak{A})$ iff $\varphi|_{\mathfrak{A}_\sigma} \equiv 0$ for every $\sigma \in \widehat{G}$, $\sigma \neq \iota$, so that we have affine weakly-* bicontinuous homeomorphisms

$$\begin{aligned} \mathcal{S}_G(\mathfrak{A}) \to \mathcal{S}_G(\mathfrak{A}_o) &\to \mathcal{S}(\mathfrak{A}^G) \\ \varphi \mapsto \quad \varphi|_{\mathfrak{A}_o} &\mapsto \varphi|_{\mathfrak{A}^G} \\ E_G^{\mathrm{t}}(f) \hookleftarrow E_G^{\mathrm{t}}(f)|_{\mathfrak{A}_o} &\hookleftarrow f\,. \end{aligned}$$

In particular, the previous homeomorphisms induce a bijective correspondence among $\mathcal{E}_G(\mathfrak{A})$, $\mathcal{E}_G(\mathfrak{A}_o)$ and $\mathcal{P}(\mathfrak{A}^G)$, where

$$\mathcal{P}(\mathfrak{A}^G) := \{f \in \mathcal{S}(\mathfrak{A}^G) \colon f = tf_1 + (1-t)f_2, t \in (0,1), f_i \in \mathcal{S}(\mathfrak{A}^G) \Rightarrow f_1 = f_2 = f\}$$

is the family of *pure* states of $\mathfrak{A}^G$.
At this stage, if we forget the topology inherited from $\mathfrak{A}$, the algebraic layer $\mathfrak{A}_o$ is merely a unital *-algebra containing $\mathfrak{A}^G$. The following result shows that $\mathfrak{A}_o$ is bounded, hence admitting an enveloping C^*-algebra $C^*(\mathfrak{A}_o)$, by the previous lemma.

Proposition II.5.2
Let $(\mathfrak{A}, G, \alpha)$ be a C^*-system, where G is compact and abelian. Then, the algebraic layer $\mathfrak{A}_o = \dot{+}_{\sigma \in \widehat{G}} \mathfrak{A}_\sigma$ is a bounded *-algebra.

Proof.
Let $a := \sum_{\sigma \in \widehat{G}} a_\sigma \in \mathfrak{A}_o$, where $a_\sigma \in \mathfrak{A}_\sigma$ is zero for all but finitely many $\sigma \in \widehat{G}$. We need to find $C_a > 0$ (only depending on a) such that $C_a \mathbb{1}_{\mathfrak{A}} - a^*a \in \sum \mathfrak{A}_o^2$. Let $S := \{\sigma \in \widehat{G} \colon a_\sigma \neq 0\}$ be the support of a and fix a total ordering on it. If $\mathbb{Z}_2^{|S|} := \{\mathbf{x} = (x_1, \ldots, x_{|S|}) \colon x_k \in \{0,1\}\}$

$$\begin{aligned} a^*a = \sum_{\sigma,\sigma' \in S} a_\sigma^* a_{\sigma'} \leq_{\mathfrak{A}_o} \sum_{\mathbf{x} \in \mathbb{Z}_2^{|S|}} \left(\sum_{\sigma \in S} (-1)^{x_\sigma} a_\sigma\right)^* \left(\sum_{\sigma' \in S} (-1)^{x_{\sigma'}} a_{\sigma'}\right) = \\ = \sum_{\sigma,\sigma' \in S} \sum_{\mathbf{x} \in \mathbb{Z}_2^{|S|}} (-1)^{x_\sigma + x_{\sigma'}} a_\sigma^* a_{\sigma'} = \sum_{\sigma,\sigma' \in S} \left(|\{x_\sigma = x_{\sigma'}\}| - |\{x_\sigma \neq x_{\sigma'}\}|\right) a_\sigma^* a_{\sigma'} = \\ = 2^{|S|} \sum_{\sigma \in S} a_\sigma^* a_\sigma + \sum_{\sigma \neq \sigma'} (2^{|S|-1} - 2^{|S|-1}) a_\sigma^* a_{\sigma'} = 2^{|S|} \sum_{\sigma \in S} a_\sigma^* a_\sigma = 2^{|S|} E_G(a^*a)\,. \end{aligned}$$

Now, $E_G(a^*a) \in \mathfrak{A}^G$ hence there exists $\sqrt{\|E_G(a^*a)\|_{\mathfrak{A}} - E_G(a^*a)} \in C^*(E_G(a^*a)) \subset \mathfrak{A}^G \subset \mathfrak{A}_o$. It follows that $a^*a \leq_{\mathfrak{A}_o} 2^{|S|} E_G(a^*a) \leq_{\mathfrak{A}_o} 2^{|S|} \|E_G(a^*a)\| \mathbb{1}_{\mathfrak{A}}$. □

Thanks to the faithfulness of E_G, we are now ready to show that $C^*(\mathfrak{A}_o)$ is naturally isomorphic to $\mathfrak{A}$, thus establishing a 1-1 correspondence between $\mathsf{Rep}(\mathfrak{A}_o)$ and $\mathsf{Rep}(\mathfrak{A})$, as well as between $\mathcal{S}(\mathfrak{A}_o)$ and $\mathcal{S}(\mathfrak{A})$. Preliminarily, notice that $\mathfrak{A}_o$ is *-semisimple, i.e. the two-sided *-ideal $\mathcal{N} = \{\|a\|_{\mathrm{u},o} = 0\} = \{0\}$: it suffices to take the restriction to $\mathfrak{A}_o$ of the universal GNS representation π_{u} of the C^*-algebra $\mathfrak{A}$, since $\pi_{\mathrm{u}}(a) = 0$ implies $a = 0$.

Proposition II.5.3
The C^*-norms $\|\cdot\|_{\mathrm{u}}$ and $\|\cdot\|_{\mathfrak{A}}$ coincide on $\mathfrak{A}_o$, whence $C^*(\mathfrak{A}_o) = \mathfrak{A}$. In particular, there exist 1-1 correspondences

- between $\mathsf{Rep}(\mathfrak{A}_o)$ and $\mathsf{Rep}(\mathfrak{A})$

- between $\mathcal{S}(\mathfrak{A}_o)$ and $\mathcal{S}(\mathfrak{A})$

Proof.
Observe that $\|\cdot\|_{\mathrm{u}}$ is compatible w.r.t. the action α i.e. α extends to an action $\widetilde{\alpha}$ on $C^*(\mathfrak{A}_o)$. Indeed, on $\mathfrak{A}_o$ we have

$$\|\alpha_g(\cdot)\|_{\mathrm{u}} = \sup_{\pi\in\mathsf{Rep}(\mathfrak{A}_o)} \|(\pi\circ\alpha_g)(\cdot)\| = \sup_{\pi\in\mathsf{Rep}(\mathfrak{A}_o)} \|\pi(\cdot)\| = \|\cdot\|_{\mathrm{u}}, \quad g\in G\,.$$

Furthermore, the maps $G\ni g\mapsto \widetilde{\alpha}_g(a)\in C^*(\mathfrak{A}_o)$ are $\|\cdot\|_{\mathrm{u}}$-continuous for each $a\in C^*(\mathfrak{A}_o)$. Indeed, for $\varepsilon>0$, choose $a_\varepsilon\in\mathfrak{A}_o$ such that $\|a-a_\varepsilon\|_{\mathrm{u}}\leq\varepsilon$, and note that, for some finite $n=n_{a,\varepsilon}$,

$$a_\varepsilon=\sum_{j=1}^n a_j\,,\quad a_j\text{'s homogeneous}\,.$$

By applying a standard 2ε-argument, we get

$$\begin{aligned}\|\widetilde{\alpha}_g(a)-a\|_{\mathrm{u}} \leq&\|\widetilde{\alpha}_g(a-a_\varepsilon)\|_{\mathrm{u}}+\|\alpha_g(a_\varepsilon)-a_\varepsilon\|_{\mathrm{u}}+\|a_\varepsilon-a\|_{\mathrm{u}}\\ \leq&2\varepsilon+\Big\|\sum_{j=1}^n\big((\partial a_j)(g)-1\big)a_j\Big\|_{\mathrm{u}}\end{aligned}$$

where ∂a_j is the degree of $a_j\in\mathfrak{A}_o$, $j=1,\dots,n$. Taking the limsup on both members, we obtain

$$\limsup_{g\to e_G}\|\widetilde{\alpha}_g(a)-a\|_{\mathrm{u}}\leq 2\varepsilon\,,$$

and the result is reached as $\varepsilon>0$ is arbitrary. In conclusion, $(C^*(\mathfrak{A}_o),G,\widetilde{\alpha})$ is a C^*-system. We now prove that its fixed point subalgebra $C^*(\mathfrak{A}_o)^G$ coincides exactly with $\mathfrak{A}^G$. By uniqueness of the C^*-norm on $\mathfrak{A}^G$, $\|\cdot\|_{\mathrm{u}}=\|\cdot\|_{\mathfrak{A}}$ on $\mathfrak{A}^G$. On the one hand, the map $E^\alpha\colon(\mathfrak{A}_o,\|\cdot\|_{\mathrm{u}})\twoheadrightarrow(\mathfrak{A}^G,\|\cdot\|_{\mathrm{u}}=\|\cdot\|_{\mathfrak{A}})$ is contractive since $\|E^\alpha(\cdot)\|_{\mathfrak{A}}\leq\|\cdot\|_{\mathfrak{A}}\leq\|\cdot\|_{\mathrm{u}}$ therefore it uniquely extends to a contraction from $C^*(\mathfrak{A}_o)$ onto $\mathfrak{A}^G$. On the other hand, since the expectation $E^{\widetilde{\alpha}}\colon C^*(\mathfrak{A}_o)\twoheadrightarrow C^*(\mathfrak{A}_o)^G$ is such that $E^{\widetilde{\alpha}}|_{\mathfrak{A}_o}=E^\alpha$, $E^{\widetilde{\alpha}}$ must be the unique bounded extension of E^α. It follows that $C^*(\mathfrak{A}_o)^G=\mathfrak{A}^G$.
By faithfulness of $E^{\widetilde{\alpha}}$ on $C^*(\mathfrak{A}_o)$, $\{\varphi\circ E^{\widetilde{\alpha}}\colon\varphi\in\mathcal{S}(\mathfrak{A}^G)\}$ is point separating for $C^*(\mathfrak{A}_o)$, and analogously for E^α on $\mathfrak{A}$, whence

$$\|\cdot\|_{\mathrm{u}}=\sup_{\varphi\in\mathcal{S}(\mathfrak{A}^G)}\|\pi_{\varphi\circ E^{\widetilde{\alpha}}}(\cdot)\|\text{ on }C^*(\mathfrak{A}_o)$$

$$\|\cdot\|_{\mathfrak{A}}=\sup_{\varphi\in\mathcal{S}(\mathfrak{A}^G)}\|\pi_{\varphi\circ E^{\alpha}}(\cdot)\|\text{ on }\mathfrak{A}$$

In particular, $\|\cdot\|_{\mathrm{u}}=\|\cdot\|_{\mathfrak{A}}$ on $\mathfrak{A}_o$. By definition of $\|\cdot\|_{\mathrm{u}}$, it follows that for every $\pi\in\mathsf{Rep}(\mathfrak{A}_o)$ we have

$$\|\pi(\cdot)\|\leq\|\cdot\|_{\mathrm{u}}=\|\cdot\|_{\mathfrak{A}}$$

on $\mathfrak{A}_o$. In other words, every representation of $\mathfrak{A}_o$ is contractive w.r.t. $\|\cdot\|_{\mathfrak{A}}$, so that there exists a 1-1 correspondence between $\mathsf{Rep}(\mathfrak{A}_o)$ and $\mathsf{Rep}(\mathfrak{A})$. In particular, the GNS representation of $\mathfrak{A}_o$ associated to some $\varphi\in\mathcal{S}(\mathfrak{A}_o)$ is contractive, or equivalently by Proposition II.3.2, φ is contractive. In other words, there exists a bijection between $\mathcal{S}(\mathfrak{A}_o)$ and $\mathcal{S}(\mathfrak{A})$. □

II.6. Bicharacters on groups

In addition to two C^*-systems $(\mathfrak{A}, G, \alpha)$ and $(\mathfrak{B}, H, \beta)$, the forthcoming construction of their twisted tensor product will heavily need the presence of a *bicharacter* on their Pontryagin duals $\widehat{G}, \widehat{H}$. In general, given two locally compact (Hausdorff) groups $\mathcal{G}, \mathcal{H}$, a bicharacter is a mapping $u\colon \mathcal{G} \times \mathcal{H} \to \mathbb{T}$ s.t. $u(g, \cdot) \in \mathsf{Hom}(\mathcal{H}, \mathbb{T})$ for every $g \in \mathcal{G}$ and $u(\cdot, h) \in \mathsf{Hom}(\mathcal{G}, \mathbb{T})$ for every $h \in \mathcal{H}$ (here, Hom stands for *continuous* group homomorphisms). Observe that $u_l\colon g \mapsto u(g, \cdot)$ is a homomorphism from $\mathcal{G}$ to $\mathsf{Hom}(\mathcal{H}, \mathbb{T})$, and similarly $u_r\colon h \mapsto u(\cdot, h)$ is a homomorphism from $\mathcal{H}$ to $\mathsf{Hom}(\mathcal{G}, \mathbb{T})$. Following the notation in [52], we will write $\mathsf{B}(\mathcal{G}, \mathcal{H}) := \{u\colon \mathcal{G} \times \mathcal{H} \to \mathbb{T}\colon u \text{ bicharacter}\}$ and $\mathsf{B}(\mathcal{G}) := \mathsf{B}(\mathcal{G}, \mathcal{G})$. The *left* and *right radicals* of $u \in \mathsf{B}(\mathcal{G}, \mathcal{H})$ are respectively defined as

$$\mathrm{Rad}_l(u) := \{g \in \mathcal{G}\colon u(g, h) = 1\, \forall h \in \mathcal{H}\} = \ker(u_l)$$

$$\mathrm{Rad}_r(u) := \{h \in \mathcal{H}\colon u(g, h) = 1\, \forall g \in \mathcal{G}\} = \ker(u_r)$$

Notice that they are closed, normal subgroups of $\mathcal{G}$ and $\mathcal{H}$, respectively. In particular, u is said to be *non-degenerate* if $\mathrm{Rad}_l(u) = (e_\mathcal{G})$ and $\mathrm{Rad}_r(u) = (e_\mathcal{H})$, *degenerate* otherwise. Every degenerate $u \in \mathsf{B}(\mathcal{G}, \mathcal{H})$ clearly induces a non-degenerate one. Indeed, if $L \le \mathrm{Rad}_l(u)$ and $R \le \mathrm{Rad}_r(u)$ are closed and normal respectively in $\mathcal{G}$ and $\mathcal{H}$, u *passes to the quotients by* L *and* R, i.e.

$$u_{L,R}\colon \mathcal{G}/L \times \mathcal{H}/R \to \mathbb{T}$$
$$(g\, L, h\, R) \mapsto u(g, h)$$

is a well-defined element of $\mathsf{B}(\mathcal{G}/L, \mathcal{H}/R)$, where

$$\mathrm{Rad}_l(u_{L,R}) = \mathrm{Rad}_l(u)/L, \quad \mathrm{Rad}_r(u_{L,R}) = \mathrm{Rad}_r(u)/R\,.$$

In particular, if $L := \mathrm{Rad}_l(u)$ and $R := \mathrm{Rad}_r(u)$, $u_{\mathrm{nd}} := u_{L,R} \in \mathsf{B}(\mathcal{G}/L, \mathcal{H}/R)$ is non-degenerate.

Let $\mathcal{G}$ be a locally compact Hausdorff group. A bicharacter $u \in \mathsf{B}(\mathcal{G})$ is *symmetric* if

$$u(g, g') = u(g', g), \quad g, g' \in \mathcal{G}\,.$$

Let $\mathsf{S}(\mathcal{G}) := \{u \in \mathsf{B}(\mathcal{G})\colon u \text{ symmetric}\}$. Similarly, u is said to be *anti-symmetric* (or *skew-symmetric*) if

$$u(g, g') = \overline{u(g', g)}, \quad g, g' \in \mathcal{G}\,.$$

Let $\mathsf{A}(\mathcal{G}) := \{u \in \mathsf{B}(\mathcal{G})\colon u \text{ anti-symmetric}\}$. Notice that if either $u \in \mathsf{S}(\mathcal{G})$ or $u \in \mathsf{A}(\mathcal{G})$, then $\mathrm{Rad}_l(u) = \mathrm{Rad}_r(u)$. In these two cases, we will simply call them *radical* of u, denoted by $\mathrm{Rad}(u)$. To every $u \in \mathsf{B}(\mathcal{G})$, we can always associate

- a symmetric bicharacter $u_\mathsf{S} \in \mathsf{S}(\mathcal{G})$, defined by $u_\mathsf{S}(g, g') := u(g, g')u(g', g)$ $(g, g' \in \mathcal{G})$
- an anti-symmetric bicharacter $u_\mathsf{A} \in \mathsf{A}(\mathcal{G})$, defined by $u_\mathsf{A}(g, g') := u(g, g')\overline{u(g', g)}$ $(g, g' \in \mathcal{G})$
- a quadratic form $Q_u\colon \mathcal{G} \to \mathbb{T}$ (i.e. $Q_u(g) = Q_u(g^{-1})$, $g \in \mathcal{G}$) given by $Q_u(g) := u(g, g)$, $g \in \mathcal{G}$

yielding a polarization identity $Q_u(gg')\overline{Q_u(g)}\,\overline{Q(g')} = u_\mathsf{S}(g, g')$, $g, g' \in \mathcal{G}$, whence $Q_u \in \mathsf{Hom}(\mathcal{G}, \mathbb{T})$ if and only if $u_\mathsf{S} \equiv 1$ (i.e. $u \in \mathsf{A}(\mathcal{G})$). In such a case, $\mathrm{im}(Q_u) \subseteq \{\pm 1\} \cong \mathbb{Z}_2$ and the Q_u-*isotropy group*

$$\Delta_+ := \ker(Q_u) = Q_u^{-1}(\{1\}) = \{g \in \mathcal{G}\colon u(g, g) = 1\}$$

is a normal closed subgroup of $\mathcal{G}$ of index $|\mathcal{G} : \Delta_+| \leq 2$ and containing both $\mathrm{Rad}(u)$ and $\{g^2 \colon g \in \mathcal{G}\}$. If $|\mathcal{G} : \Delta_+| = 1$, i.e. $u(g,g) = 1$ for every $g \in \mathcal{G}$, u is said to be *alternating*, otherwise $|\mathcal{G} : \Delta_+| = 2$ and the unique non-trivial coset of Δ_+ in $\mathcal{G}$ is

$$\Delta_- := Q_u^{-1}(\{-1\}) = \{g \in \mathcal{G} \colon u(g,g) = -1\}$$

Since we are interested in the case $\mathcal{G} = \widehat{G}, \mathcal{H} = \widehat{H}$, where G, H are compact abelian groups, we now focus on bicharacters on discrete abelian groups (or equivalently, $\mathbb{Z}$-modules). In such a case, $\mathsf{B}(\widehat{G}, \widehat{H}) \cong \mathsf{Hom}(\widehat{G}, H) \cong \mathsf{Hom}(\widehat{H}, G) \cong \mathsf{Hom}(\widehat{G} \otimes_{\mathbb{Z}} \widehat{H}, \mathbb{T})$. Moreover, if $G = H$ is finite (hence isomorphic to a direct product of cyclic groups of prime power order, and in particular self-dual), the polarization identity above induces a surjection

$$\begin{aligned} \{Q \colon \widehat{G} \to \mathbb{T} \colon Q \text{ quadratic form}\} &\twoheadrightarrow \mathsf{S}(\widehat{G}) \\ Q &\mapsto [(\sigma,\tau) \mapsto Q(\sigma\tau)\overline{Q(\sigma)}\,\overline{Q(\tau)}] \end{aligned}$$

so that every symmetric bicharacter on $\widehat{G}$ comes from some quadratic form. This mapping is 1-1 when $\widehat{G}$ has odd order, with inverse mapping given by $\mathsf{S}(\widehat{G}) \ni u \mapsto Q_u$. As concerns $\mathsf{A}(\widehat{G})$, Scheunert classified them in Section 5 of [72], p. 715-716, in the case when $\widehat{G}$ is a finitely generated abelian group (here, anti-symmetric bicharacters are called *commutation factors*, as they are used to define commutation relations on a u-Lie algebra). We report here its result. We use the convention that $\mathbb{Z}_0 = \mathbb{Z}/0\mathbb{Z} = \mathbb{Z}$ and $\gcd(0,n) = n$ for any $n \geq 0$. Moreover, for every $i = 1, \ldots, t$, let $1_i := [1]_{n_i} \in \mathbb{Z}_{n_i}$.

Proposition II.6.1 (Scheunert, [72])
Let $\widehat{G}$ be finitely generated. If $\widehat{G} \cong \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_t}$ is a decomposition of $\widehat{G}$ into cyclic groups, then each $u \in \mathsf{A}(\widehat{G})$ is uniquely determined by an Hermitian matrix

$$\mathcal{U} = (u_{ij})_{i,j=1}^t := \big(u(1_i, 1_j)\big)_{i,j=1}^t \in M_t(\mathbb{T})$$

satisfying

(1) $u_{ij}^{n_{ij}} = 1$, $n_{ij} := \gcd(n_i, n_j)$

(2) $u_{ii} = \begin{cases} \pm 1 \text{ if } n_i \text{ even} \\ 1 \text{ if } n_i \text{ odd} \end{cases}$

for every $i, j = 1, \ldots, t$. Precisely,

$$u\left(\sum_{i=1}^t \sigma_i, \sum_{j=1}^t \tau_j\right) = \prod_{k=1}^t u_{kk}^{\sigma_k \tau_k} \prod_{1 \leq i < j \leq t} u_{ij}^{\sigma_i \tau_j - \sigma_j \tau_i} \qquad (\sigma_i, \tau_j \in \mathbb{Z})$$

In particular,

- Δ_+ is generated by a set $\mathcal{S}_{\Delta_+}$ s.t. $|\mathcal{S}_{\Delta_+}| \leq |\widehat{G} \colon \Delta_+|\, t$ (Schreier lemma)
- $|\widehat{G} \colon \Delta_+| = 1$ (i.e. $\Delta_+ = \widehat{G}$, i.e. u is *alternating*) if and only if u is induced by some $\gamma \in \mathsf{B}(\widehat{G})$, i.e.
$$u(\sigma,\tau) = h_\gamma(\sigma,\tau) = \gamma(\sigma,\tau)\overline{\gamma(\tau,\sigma)}, \quad \sigma,\tau \in \widehat{G}$$
- $|\widehat{G} \colon \Delta_+| = 2$ (i.e. u is *non-alternating*) if and only if $\mathcal{A} := \{i \in \{1,\ldots,t\} \colon u_{ii} = -1\} \neq \varnothing$. In such a case, n_i is even for some $i = 1, \ldots, t$ and
$$\Delta_+ = \{(\sigma_1, \ldots, \sigma_t) \colon |\{i \in \mathcal{A} \colon \sigma_i \text{ odd}\}| \text{ is even}\}$$
$$\Delta_- = \{(\sigma_1, \ldots, \sigma_t) \colon |\{i \in \mathcal{A} \colon \sigma_i \text{ odd}\}| \text{ is odd}\}\,.$$

The following table shows the explicit form of $\mathsf{B}(\widehat{G})$, $\mathsf{A}(\widehat{G})$ and $\mathsf{S}(\widehat{G})$ for notable compact abelian groups G. In this table only, we will denote the finite cyclic groups as $\mathbb{Z}/n\mathbb{Z}$ ($n \geq 2$) in order to distinguish them from the (additive) p-adic integers group $\mathbb{Z}_p$, p being a prime. It is the profinite group arising from the inverse limit of the canonical surjections $\mathbb{Z}/p^n\mathbb{Z} \twoheadrightarrow \mathbb{Z}/p^{n-1}\mathbb{Z}$, and its Pontryagin dual is the Prüfer p-group $\mathbb{Z}(p^\infty)$, isomorphic to the quotient $\mathbb{Q}_p/\mathbb{Z}_p$ of rational p-adics by integer ones. Similarly, $\hat{\mathbb{Z}}$ is the profinite completion of $\mathbb{Z}$, the inverse limit of the canonical surjections $\mathbb{Z}/m\mathbb{Z} \twoheadrightarrow \mathbb{Z}/n\mathbb{Z}$ where $n|m$. $\mathbf{Bohr}(\mathbb{R})$ and $\mathbb{R}_{\mathrm{d}}$ are respectively the Bohr compactification and the discretization of $\mathbb{R}$. They are in duality, as $\mathbf{Bohr}(\mathbb{Z})$ and $\mathbb{T}_{\mathrm{d}}$. Lastly, the additive group structure underlying the field $\mathbb{Q}$ can be viewed as the (discrete) dual of the compact abelian group $\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}$, where $\mathbb{A}_{\mathbb{Q}}$ is the adele ring of $\mathbb{Q}$ (or better said, the underlying additive group). $\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}$ has also a description of direct limit of morphisms $\rho_{nm} \colon \mathbb{T} \hookrightarrow \mathbb{T}$, $\rho_{nm}(z) := z^{(n^{m-n})}$, $m \geq n$, $z \in \mathbb{T}$ (for a reference, see [94], p. 404). The last column of the following table, concerning the decomposition of the group $\mathsf{B}(\widehat{G})$, is mostly attributable to Kleppner (see [52]).

G compact	$\widehat{G}$ discrete	$\mathsf{B}(\widehat{G})$	$\mathsf{A}(\widehat{G})$	$\mathsf{S}(\widehat{G})$	Decomposition of $\mathsf{B}(\widehat{G})$
$(\mathbb{Z}/n\mathbb{Z})^N$	$(\mathbb{Z}/n\mathbb{Z})^N$	$(\mathbf{x},\mathbf{y}) \mapsto e^{i\frac{2\pi}{n}\mathbf{x}^t M\mathbf{y}}$ $M \in M_N(\mathbb{Z}/n\mathbb{Z})$	$M \in M_N(\mathbb{Z}/n\mathbb{Z})$ anti-symmetric	$M \in M_N(\mathbb{Z}/n\mathbb{Z})$ symmetric	$\mathsf{A}(\widehat{G}) \times \mathsf{S}(\widehat{G})$ if n odd $\mathsf{A}(\widehat{G})\mathsf{S}(\widehat{G})$ if n even
$\mathbb{T}^N$ $\cong SO_2(\mathbb{C})^N \cong (\mathbb{R}/\mathbb{Z})^N$	$\mathbb{Z}^N$	$(\mathbf{x},\mathbf{y}) \mapsto e^{i2\pi\mathbf{x}^t M\mathbf{y}}$ $M \in M_N([0,1))$	$M \in M_N([0,1))$ anti-symmetric	$M \in M_N([0,1))$ symmetric	$\mathsf{A}(\widehat{G})\mathsf{S}(\widehat{G})$
$\mathbf{Bohr}(\mathbb{R})$	$\mathbb{R}_{\mathrm{d}}$	$(x,y) \mapsto e^{i2\pi x(\mathbf{r})y}$ $\mathbf{r} \in \mathbf{Bohr}(\mathbb{R})$	$x(\mathbf{r})y = -y(\mathbf{r})x$ $x,y \in \mathbb{R}_{\mathrm{d}}, \mathbf{r} \in \mathbf{Bohr}(\mathbb{R})$	$x(\mathbf{r})y = y(\mathbf{r})x$ $x,y \in \mathbb{R}_{\mathrm{d}}, \mathbf{r} \in \mathbf{Bohr}(\mathbb{R})$	$\mathsf{A}(\widehat{G}) \times \mathsf{S}(\widehat{G})$
$\mathbf{Bohr}(\mathbb{Z})$	$\mathbb{T}_{\mathrm{d}}$	$(x,y) \mapsto e^{i2\pi x(\mathbf{z})y}$ $\mathbf{z} \in \mathbf{Bohr}(\mathbb{Z})$	$x(\mathbf{z})y = -y(\mathbf{z})x$ $x,y \in \mathbb{T}_{\mathrm{d}}, \mathbf{z} \in \mathbf{Bohr}(\mathbb{Z})$	$x(\mathbf{z})y = y(\mathbf{z})x$ $x,y \in \mathbb{T}_{\mathrm{d}}, \mathbf{z} \in \mathbf{Bohr}(\mathbb{Z})$	$\mathsf{A}(\widehat{G}) \times \mathsf{S}(\widehat{G})$
$\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}$ $\cong \varinjlim \mathbb{T}$	$\mathbb{Q}$	$(x,y) \mapsto e^{i2\pi x(\mathbf{a})y}$ $\mathbf{a} \in \mathbb{A}_{\mathbb{Q}}/\mathbb{Q}$	$x(\mathbf{a})y = -y(\mathbf{a})x$ $x,y \in \mathbb{Q}, \mathbf{a} \in \mathbb{A}_{\mathbb{Q}}/\mathbb{Q}$	$x(\mathbf{a})y = y(\mathbf{a})x$ $x,y \in \mathbb{Q}, \mathbf{a} \in \mathbb{A}_{\mathbb{Q}}/\mathbb{Q}$	$\mathsf{A}(\widehat{G}) \times \mathsf{S}(\widehat{G})$
$\mathbb{Z}_p := \varprojlim \mathbb{Z}/p^n\mathbb{Z}$ $\cong \mathsf{End}(\mathbb{Z}(p^\infty))$	$\mathbb{Z}(p^\infty) := \varinjlim \mathbb{Z}/p^n\mathbb{Z}$ $\cong \mathbb{Q}_p/\mathbb{Z}_p$	$\mathbb{1}$	$\mathbb{1}$	$\mathbb{1}$	$\mathsf{A}(\widehat{G}) \times \mathsf{S}(\widehat{G})$
$\hat{\mathbb{Z}} := \varprojlim \mathbb{Z}/n\mathbb{Z}$ $\cong \prod_p \mathbb{Z}_p$	$\mathbb{Q}/\mathbb{Z}$	$\mathbb{1}$	$\mathbb{1}$	$\mathbb{1}$	$\mathsf{A}(\widehat{G}) \times \mathsf{S}(\widehat{G})$

Remark II.6.2
If $\mathcal{G}$ and $\mathcal{H}$ are discrete (possibly, non-abelian) groups, then the derived/commutator subgroups $\mathcal{G}^{(1)} := [\mathcal{G}, \mathcal{G}]$ and $\mathcal{H}^{(1)} := [\mathcal{H}, \mathcal{H}]$ are normal in $\mathcal{G}$ and $\mathcal{H}$ respectively, as well as evidently closed. In addition, by commutativity of $\mathbb{T}$, $\mathcal{G}^{(1)} \leq \mathrm{Rad}_l(u)$ and $\mathcal{H}^{(1)} \leq \mathrm{Rad}_r(u)$, so that $u \in \mathsf{B}(\mathcal{G}, \mathcal{H})$ descends to the abelianizations $\mathcal{G}^{\mathrm{ab}} := \mathcal{G}/\mathcal{G}^{(1)}$ and $\mathcal{H}^{\mathrm{ab}} := \mathcal{H}/\mathcal{H}^{(1)}$, which are still discrete (since the quotient mapping π^{ab} is open). Let $u_{\mathrm{ab}} := u_{\mathcal{G}^{(1)}, \mathcal{H}^{(1)}} \in \mathsf{B}(\mathcal{G}^{\mathrm{ab}}, \mathcal{H}^{\mathrm{ab}})$. Notice that $\widehat{\mathcal{G}^{\mathrm{ab}}} \cong \mathsf{Hom}(\mathcal{G}, \mathbb{T})$ and $\widehat{\mathcal{H}^{\mathrm{ab}}} \cong \mathsf{Hom}(\mathcal{H}, \mathbb{T})$ are compact abelian groups.

II.7. Algebraic twisted tensor products of graded C^*-algebras

We are finally ready to construct the algebraic twisted tensor product of two graded C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$. For such a purpose, we start from C^*-systems $(\mathfrak{A}, G, \alpha)$ and $(\mathfrak{B}, H, \beta)$, where G and H are compact abelian groups, and a bicharacter $u : \widehat{G} \times \widehat{H} \to \mathbb{T}$. The reader is referred to [57] and [70] for a more abstract/categorical (but less constructive) approach to twisted tensor products, arising from (continuous, right) coactions of locally compact C^*-quantum groups (in the sense of Woronowicz, see [81]). In both papers, the authors directly define new C^*-algebras via representations on Hilbert spaces, without analyzing their algebraic properties and/or their (invariant) state spaces further. In any case, they will correspond respectively to the minimal and maximal C^*-completions of our algebraic twisted tensor product, as we will see in Section II.17.
For C^*-systems $(\mathfrak{A}, G, \alpha)$ and $(\mathfrak{B}, H, \beta)$, we first consider the algebraic layers $\mathfrak{A}_o$ and $\mathfrak{B}_o$ generated by homogeneous elements, and given for $\mathfrak{A}$ by

$$\mathfrak{A}_o = \left\{ \sum_{\sigma \in F} a_\sigma \colon a_\sigma \text{ homogeneous}, \ F \subset \widehat{G} \text{ finite} \right\},$$

and analogously for $\mathfrak{B}_o \subset \mathfrak{B}$. After fixing a bicharacter

$$\widehat{G} \times \widehat{H} \ni (\sigma, \tau) \mapsto u(\sigma, \tau) \in \mathbb{T},$$

we can construct the algebraic twisted tensor product $\mathfrak{A}_o \circledast \mathfrak{B}_o$ by equipping the linear space $\mathfrak{A}_o \odot \mathfrak{B}_o$ with a *-operation and a compatible product in order to make it an involutive algebra. We shall denote the involution with *, whereas the product with a simple juxtaposition, to distinguish them from $\cdot$ and † on the usual (non-twisted) tensor product $\mathfrak{A}_o \otimes \mathfrak{B}_o$. This is done directly on homogeneous elements $a \in \mathfrak{A}_\sigma$ and $b \in \mathfrak{B}_\tau$, with the convention that we sometimes write $u(a, b) := u(\sigma, \tau)$.
Indeed, on generators $a \in \mathfrak{A}_o$ and $A \in \mathfrak{A}_\sigma$, $b \in \mathfrak{B}_\tau$ and $B \in \mathfrak{B}_o$, we put

$$(a \odot b)(A \odot B) := \overline{u(\sigma, \tau)}\big((a \odot b) \cdot (A \odot B)\big) = \overline{u(\sigma, \tau)} aA \odot bB \,. \tag{II.6}$$

One straightforwardly checks that ((II.6)) is associative and bilinear. Concerning the involution, for $a \in \mathfrak{A}_\sigma$, $b \in \mathfrak{B}_\tau$, we put

$$(a \odot b)^* := \overline{u(\sigma, \tau)}(a \odot b)^\dagger = \overline{u(\sigma, \tau)} a^* \odot b^* \,. \tag{II.7}$$

The operation in ((II.7)) is evidently involutive and $\mathbb{C}$-antilinear.

Proposition II.7.1
The linear space $\mathfrak{A}_o \odot \mathfrak{B}_o$, equipped with the involution (II.7) and product (II.6), is an involutive algebra.

Proof.
The only crucial point to check is that (II.7) and (II.6) are well defined. Indeed, we note that

$$\mathfrak{A}_o \odot \mathfrak{B}_o = \dot{+}_{\sigma\in\widehat{G},\tau\in\widehat{H}} \mathfrak{A}_\sigma \odot \mathfrak{B}_\tau \,,$$
$$\mathfrak{B}_o \odot \mathfrak{A}_o = \dot{+}_{\sigma\in\widehat{G},\tau\in\widehat{H}} \mathfrak{B}_\tau \odot \mathfrak{A}_\sigma \,.$$

On such homogeneous subspaces, define

$$\Phi_{\sigma,\tau}(a_\sigma \odot b_\tau) := u(\sigma,\tau) a_\sigma \odot b_\tau \,, \quad \Psi_{\sigma,\tau}(b_\tau \odot a_\sigma) := \overline{u(\sigma,\tau)} b_\tau \odot a_\sigma \,.$$

The above maps are all manifestly well defined, and thus uniquely extend to linear maps $\Phi : \mathfrak{A}_o \odot \mathfrak{B}_o \to \mathfrak{A}_o \odot \mathfrak{B}_o$ and $\Psi : \mathfrak{B}_o \odot \mathfrak{A}_o \to \mathfrak{B}_o \odot \mathfrak{A}_o$, respectively.
Concerning the star operation, it is clear that $* = \dagger \circ \Phi$ and, concerning the product maps $M_{\textcircled{u}}$ and $M_\otimes$,

$$M_{\textcircled{u}} = M_\otimes \circ \left(\mathrm{id}_{\mathfrak{A}_o} \otimes \Psi \otimes \mathrm{id}_{\mathfrak{B}_o}\right) .$$

Therefore, (II.7) and (II.6) are well defined. □

Definition II.7.2
We denote by $\mathfrak{A}_o \textcircled{u} \mathfrak{B}_o$ the linear space $\mathfrak{A}_o \odot \mathfrak{B}_o$ equipped with the *-operation (II.7) and the product (II.6). We name it *twisted tensor product of* $\mathfrak{A}_o$ *and* $\mathfrak{B}_o$ associated to the bicharacter u.

Here, the symbol "$\textcircled{u}$" stands for "tensor product twisted by the bicharacter u". Notice that, for elements of $\mathfrak{A}_o \textcircled{u} \mathfrak{B}_o$ of the form $\widetilde{A} := a \odot \mathbb{1}_{\mathfrak{A}}$ and $\widetilde{B} := \mathbb{1}_{\mathfrak{B}} \odot b$, $a \in \mathfrak{A}_\sigma$, $b \in \mathfrak{B}_\tau$, we incidentally get the commutation rule

$$\widetilde{A}\widetilde{B} = u(\sigma,\tau)\widetilde{B}\widetilde{A} \,,$$

which is well established for the *Fermi* tensor product (*e.g.* [49],[31],[19]), and the *rotation algebras* (*e.g.* [85]). In particular, the (algebraic) Fermi product of two $\mathbb{Z}_2$-graded C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ is defined as the involutive algebra $\mathfrak{A} \textcircled{F} \mathfrak{B}$ where

$$(a \textcircled{F} b)^* = (-1)^{\partial a \partial b} a^* \textcircled{u} b^*, \; (A \textcircled{F} b)(a \textcircled{F} B) = (-1)^{\partial a \partial b} Aa \textcircled{F} bB \; (a, A \in \mathfrak{A}, b, B \in \mathfrak{B}, a, b \text{ hom.})$$

having identified $\mathbb{Z}_2$ with $\{0, 1\}$.
Recall that for any non-empty subset $X \subseteq \widehat{G}$, the *annihilator* of X is defined as

$$X^\perp := \{g \in G \colon \sigma(g) = 1, \, \sigma \in X\} \,.$$

Obviously, $X^\perp = \langle X \rangle^\perp$ so that if we have an action $G \overset{\alpha}{\curvearrowright} \mathfrak{A}$, $\mathrm{spec}_o(\alpha)^\perp = \mathrm{spec}(\alpha)^\perp = \ker(\alpha)$. Consider the C^*-system $\big(\mathfrak{A}, \widetilde{G}, \widetilde{\alpha}\big)$, where $\widetilde{G} := G/\mathrm{spec}(\alpha)^\perp$ (with Pontryagin dual $\mathrm{spec}(\alpha)$) and $\widetilde{\alpha}$ is the faithful *quotient action* of $\widetilde{G}$ on $\mathfrak{A}$. It is a matter of straightforward computation to prove the following

Proposition II.7.3
Let $(\mathfrak{A}, G, \alpha)$ and $(\mathfrak{B}, H, \beta)$ be C^*-systems, together with a bicharacter u. Then, the associated twisted tensor product coincides with that associated to the quotient C^*-systems $\big(\mathfrak{A}, \widetilde{G}, \widetilde{\alpha}\big)$, $\big(\mathfrak{B}, \widetilde{H}, \widetilde{\beta}\big)$, and the bicharacter $u|_{\mathrm{spec}(\alpha)\times\mathrm{spec}(\beta)}$.

A dual result is connected to the structure of the involved bicharacter u. Indeed, consider the two subgroups $L := \mathrm{Rad}_l(u) \subset \widehat{G}$ and $R := \mathrm{Rad}_r(u) \subset \widehat{H}$. As seen in the previous section, the bicharacter u passes to quotients $\widehat{G}/L$ and $\widehat{H}/R$, namely

$$u_{\mathrm{nd}}(\sigma l, \tau r) := u(\sigma, \tau) \,, \quad \sigma \in \widehat{G}, \; l \in L, \; \tau \in \widehat{H}, \; r \in R \,,$$

is a well-defined, non-degenerate bicharacter. Also, we recall that

$$\begin{cases} \widehat{L} \cong G/L^\perp \\ L^\perp \cong \widehat{\widehat{G}/L} \end{cases} \qquad \begin{cases} \widehat{R} \cong G/R^\perp \\ R^\perp \cong \widehat{\widehat{H}/R} \end{cases}$$

If two C^*-systems as above are given, $L^\perp$ acts on $\mathfrak{A}$ via $\alpha|_{L^\perp}$ whereas $R^\perp$ acts on $\mathfrak{B}$ via $\beta|_{R^\perp}$. Let $\mathfrak{A}_G$ and $\mathfrak{A}_L$ be the same C^*-algebra $\mathfrak{A}$, the former endowed with the $\widehat{G}$-grading induced by α, the latter with the $\widehat{G}/L$-grading induced by $\alpha|_{L^\perp}$. Then, the $\widehat{G}/L$-grading "packs up" the original $\widehat{G}$-grading by exploiting the cosets of $L = \mathrm{Rad}_l(u)$. Precisely, if $\mathcal{C} := \{C \subset \widehat{G} \colon C \text{ coset of } L\}$,

$$(\mathfrak{A}_L)_C = \bigoplus_{\sigma \in C} (\mathfrak{A}_G)_\sigma, \quad C \in \mathcal{C}$$

so that the algebraic layer of $\mathfrak{A}_L$ is $(\mathfrak{A}_L)_o = \dot{+}_{C\in\mathcal{C}} (\mathfrak{A}_L)_C = \dot{+}_{C\in\mathcal{C}} \left[\bigoplus_{\sigma\in C} (\mathfrak{A}_G)_\sigma\right]$. If we ignore the closure in $\mathfrak{A}$ and define

$$(\mathfrak{A}_L)_{C,o} := \dot{+}_{\sigma \in C} (\mathfrak{A}_G)_\sigma \,,$$

$$(\mathfrak{A}_L)_{oo} := \dot{+}_{C\in\mathcal{C}} (\mathfrak{A}_L)_{C,o} = \dot{+}_{C\in\mathcal{C}} \left[\dot{+}_{\sigma\in C} (\mathfrak{A}_G)_\sigma\right]$$

we immediately see that $(\mathfrak{A}_G)_o$ and $(\mathfrak{A}_L)_{oo}$ are isomorphic as $\mathbb{C}$-linear spaces. In a similar way, the $\widehat{H}/R$-grading on $\mathfrak{B}$ employs the cosets of R to rearrange the $\widehat{H}$-grading. Furthermore,

Proposition II.7.4
The identity is a *-isomorphism of involutive algebras

$$\mathrm{id} \colon (\mathfrak{A}_G)_o \textcircled{u} (\mathfrak{B}_H)_o \to (\mathfrak{A}_L)_{oo} \textcircled{u_{\mathrm{nd}}} (\mathfrak{B}_R)_{oo} \subseteq (\mathfrak{A}_L)_o \textcircled{u_{\mathrm{nd}}} (\mathfrak{B}_R)_o$$

Proof.
Let $a, A \in (\mathfrak{A}_G)_o$ (with a homogeneous) and $b, B \in (\mathfrak{B}_H)_o$ (with b homogeneous). Then,

$$(a \textcircled{u} b)^* = \overline{u(a,b)} a^* \textcircled{u} b^* = \overline{u_{\mathrm{nd}}(aR, bS)} a^* \textcircled{u_{\mathrm{nd}}} b^* = (a \textcircled{u_{\mathrm{nd}}} b)^*$$

$$(A \textcircled{u} b)(a \textcircled{u} B) = \overline{u(a,b)} Aa \textcircled{u} bB = \overline{u_{\mathrm{nd}}(aR, bS)} Aa \textcircled{u} bB = (A \textcircled{u_{\mathrm{nd}}} b)(a \textcircled{u_{\mathrm{nd}}} B) \,. \qquad \square$$

We shall come back to this construction afterwards, when C^*-completions of an algebraic twisted tensor product will be available. For the moment, notice that Proposition II.7.3 and Proposition II.7.4 can be combined to obtain a kind of "non-degenerate" twisted tensor product, where both the annihilators of the two spectra and the radicals of the bicharacter are modded out.

For future scopes, we also describe here two natural isomorphisms involving twisted tensor products: the *flip* (i.e. the isomorphism realizing the swap of the marginal algebras), and the *factoring-out* mappings.

Proposition II.7.5
The twisted tensor product $\mathfrak{A}_o \textcircled{u} \mathfrak{B}_o$ is naturally *-isomorphic to $\mathfrak{B}_o \textcircled{v} \mathfrak{A}_o$, where the bicharacter $v : \widehat{H} \times \widehat{G} \to \mathbb{T}$ is given by

$$v(b, a) := \overline{u(a,b)} \,, \quad a \in \mathfrak{A}_o \,, b \in \mathfrak{B}_o \text{ homogeneous} \,.$$

The isomorphism is given on the homogeneous elements by

$$\mathfrak{A}_\sigma \odot \mathfrak{B}_\tau \ni a \odot b \mapsto \Phi_u(a \odot b) := u(a,b) b \odot a \in \mathfrak{A}_\tau \odot \mathfrak{B}_\sigma \,.$$

Proof.
It is a matter of straightforward computation. For $a, A \in \mathfrak{A}_o$ (with a homogeneous) and $b, B \in \mathfrak{B}_o$ (with b homogeneous),

$$\begin{aligned}\Phi_u((a \mathbin{\textcircled{u}} b)^*) &= \overline{u(a,b)}\Phi_u(a^* \mathbin{\textcircled{u}} b^*) = \overline{u(a,b)}u(a^*,b^*)b^* \mathbin{\textcircled{v}} a^* = b^* \mathbin{\textcircled{v}} a^* = \\ &= v(b,a)(b \mathbin{\textcircled{v}} a)^* = (\overline{v(b,a)}b \mathbin{\textcircled{v}} a)^* = (u(a,b)b \mathbin{\textcircled{v}} a)^* = \Phi_u(a \mathbin{\textcircled{u}} b)^*\end{aligned}$$

$$\begin{aligned}\Phi_u((A \mathbin{\textcircled{u}} b)(a \mathbin{\textcircled{u}} B)) &= \overline{u(a,b)}\Phi_u(Aa \mathbin{\textcircled{u}} bB) = \overline{u(a,b)}u(Aa,bB)bB \mathbin{\textcircled{v}} Aa = \\ &= u(A,b)u(a,B)u(A,B)bB \mathbin{\textcircled{v}} Aa = u(A,b)u(a,B)\overline{v(B,A)}bB \mathbin{\textcircled{v}} Aa \\ &= u(A,b)u(a,B)(b \mathbin{\textcircled{v}} A)(B \mathbin{\textcircled{v}} a) = \Phi_u(A \mathbin{\textcircled{u}} b)\Phi_u(a \mathbin{\textcircled{u}} B)\,. \qquad \square\end{aligned}$$

Lastly, we discuss the factoring-out maps (II.1) in the case when there is an additional structure of involutive algebras and associated twists.
Let $(\mathfrak{A}_i, G_i, \alpha_i)$, $i = 1, 2$, and $(\mathfrak{B}, H, \beta)$ be three C^*-systems as above. We also fix two bicharacters $u_i : \widehat{G_i} \times \widehat{H} \to \mathbb{T}$. We then define a new C^*-system $(\mathfrak{A}, G, \alpha)$, where $\mathfrak{A} := \mathfrak{A}_1 \oplus \mathfrak{A}_2$, $G := G_1 \times G_2$, and $\alpha := \alpha_1 \oplus \alpha_2$ defined by

$$\alpha_g(a) := (\alpha_1)_{g_1}(a_1) \oplus (\alpha_2)_{g_2}(a_2)\,, \quad a = a_1 \oplus a_2, g = (g_1, g_2). \tag{II.8}$$

For the bicharacter, we put

$$u\big((\sigma_1, \sigma_2), \tau\big) := u_1(\sigma_1, \tau)u_2(\sigma_2, \tau)\,, \quad \sigma_i \in \widehat{G_i},\ i = 1, 2,\ \tau \in \widehat{H}\,. \tag{II.9}$$

An elementary checking shows that

$$\big((\mathfrak{A}_1)_o \mathbin{\textcircled{\scriptsize u_1}} \mathfrak{B}_o\big) \oplus \big((\mathfrak{A}_2)_o \mathbin{\textcircled{\scriptsize u_2}} \mathfrak{B}_o\big) \cong \big((\mathfrak{A}_1 \oplus \mathfrak{A}_2)_o \mathbin{\textcircled{u}} \mathfrak{B}_o\big)\,,$$

where the *-isomorphism is inherited by the right factoring-out map R in (II.1).
We want to point out that this isomorphism relies on the structure of $\mathrm{spec}_o(\alpha_1 \oplus \alpha_2) \subset \widehat{G_1 \times G_2} \cong \widehat{G_1} \times \widehat{G_2}$. Indeed,

$$\begin{aligned}E^{G_1 \times G_2}_{\sigma_1,\sigma_2}(a_1 \oplus a_2) &= \iint_{G_1 \times G_2} \overline{(\sigma_1,\sigma_2)(g_1,g_2)}\alpha_{(g_1,g_2)}(a_1 \oplus a_2)\mathrm{d}g_1\mathrm{d}g_2 = \\ &= \iint_{G_1 \times G_2} \overline{\sigma_1(g_1)\sigma_2(g_2)}\big((\alpha_1)_{g_1}(a_1) \oplus (\alpha_2)_{g_2}(a_2)\big)\mathrm{d}g_1\mathrm{d}g_2 = \\ &= \Big(\iint_{G_1 \times G_2} \overline{\sigma_1(g_1)\sigma_2(g_2)}(\alpha_1)_{g_1}(a_1)\mathrm{d}g_1\mathrm{d}g_2\Big) \oplus \Big(\iint_{G_1 \times G_2} \overline{\sigma_1(g_1)\sigma_2(g_2)}(\alpha_2)_{g_2}(a_2)\mathrm{d}g_1\mathrm{d}g_2\Big) = \\ &= \Big(\int_{G_1} \overline{\sigma_1(g_1)}(\alpha_1)_{g_1}(a_1)\mathrm{d}g_1 \int_{G_2} \overline{\sigma_2(g_2)}\mathrm{d}g_2\Big) \oplus \Big(\int_{G_1} \overline{\sigma_1(g_1)}\mathrm{d}g_1 \int_{G_2} \overline{\sigma_2(g_2)}(\alpha_2)_{g_2}(a_2)\mathrm{d}g_2\Big) = \\ &= \Big(\delta_{\sigma_2, e_{\widehat{G_2}}} \int_{G_1} \overline{\sigma_1(g_1)}(\alpha_1)_{g_1}(a_1)\mathrm{d}g_1\Big) \oplus \Big(\delta_{\sigma_1, e_{\widehat{G_1}}} \int_{G_2} \overline{\sigma_2(g_2)}(\alpha_2)_{g_2}(a_2)\mathrm{d}g_2\Big) = \\ &= \big(\delta_{\sigma_2, e_{\widehat{G_2}}} E_{\sigma_1}(a_1)\big) \oplus \big(\delta_{\sigma_1, e_{\widehat{G_1}}} E_{\sigma_2}(a_2)\big)\,,\end{aligned}$$

and this leads to $\mathrm{spec}_o(\alpha) \subset \widehat{G_1} \cup \widehat{G_2} \subset \widehat{G_1} \times \widehat{G_2}$.

For the convenience of the reader, we report the specular situation as well. Starting from C^*-systems $(\mathfrak{A}, G, \alpha)$, $(\mathfrak{B}_i, H_i, \beta_i)$, $i = 1, 2$, and bicharacters $u_i : \widehat{G} \times \widehat{H_i} \to \mathbb{T}$, we put $\mathfrak{B} := \mathfrak{B}_1 \oplus \mathfrak{B}_2$, $H := H_1 \times H_2$,

$$\beta_h(b) := (\beta_1)_{h_1}(b_1) \oplus (\beta_2)_{h_2}(b_2)\,, \quad b_i \in \mathfrak{B}_i,\ h_i \in H_i\,, \tag{II.10}$$

$$u\big(\sigma, (\tau_1, \tau_2)\big) := u_1(\sigma, \tau_1) u_2(\sigma, \tau_2)\,, \ \ \sigma \in \widehat{G},\ \tau_i \in \widehat{H_i},\ i = 1, 2\,. \tag{II.11}$$

The left factoring-out map L realizes the *-isomorphism

$$\big(\mathfrak{A}_o \,\textcircled{\scriptsize u_1}\, (\mathfrak{B}_1)_o\big) \oplus \big(\mathfrak{A}_o \,\textcircled{\scriptsize u_2}\, (\mathfrak{B}_2)_o\big) \cong \big(\mathfrak{A}_o \,\textcircled{\scriptsize u}\, (\mathfrak{B}_1 \oplus \mathfrak{B}_2)_o\big)\,.$$

In case two C^*-systems share the same group, there is a further construction involving the *diagonal* action of that group. Let $(\mathfrak{A}_i, G, \alpha_i)$, $i = 1, 2$, and $(\mathfrak{B}, H, \beta)$ three C^*-systems. Fix a bicharacter $u : \widehat{G} \times \widehat{H} \to \mathbb{T}$. We get a new C^*-system $(\mathfrak{A}_1 \oplus \mathfrak{A}_2, G, d^{(\alpha)})$ where the diagonal action $G \overset{d^{(\alpha)}}{\curvearrowright} \mathfrak{A}$ is defined as

$$d_g^{(\alpha)}(a) := (\alpha_1)_g(a_1) \oplus (\alpha_2)_g(a_2)\,, \quad a = a_1 \oplus a_2,\ g \in G\,. \tag{II.12}$$

As before, we can straightforwardly verify that

$$\big((\mathfrak{A}_1)_o \,\textcircled{\scriptsize u}\, \mathfrak{B}_o\big) \oplus \big((\mathfrak{A}_2)_o \,\textcircled{\scriptsize u}\, \mathfrak{B}_o\big) \cong \big((\mathfrak{A}_1 \oplus \mathfrak{A}_2)_o \,\textcircled{\scriptsize u}\, \mathfrak{B}_o\big)\,,$$

where the *-isomorphism is inherited by the right factoring-out map R in (II.1).
Similarly, the case of the left factoring-out map L is summarized as follows. We start with C^*-systems $(\mathfrak{A}, G, \alpha)$, $(\mathfrak{B}_i, H, \beta_i)$, $i = 1, 2$, and a bicharacter u. The diagonal action $H \overset{d^{(\beta)}}{\curvearrowright} \mathfrak{B}_1 \oplus \mathfrak{B}_2$ is then defined as

$$d_h^{(\beta)}(b) := (\beta_1)_h(b_1) \oplus (\beta_2)_h(b_2)\,, \quad b = b_1 \oplus b_2$$

whence we have

$$\big(\mathfrak{A}_o \,\textcircled{\scriptsize u}\, (\mathfrak{B}_1)_o\big) \oplus \big(\mathfrak{A}_o \,\textcircled{\scriptsize u}\, (\mathfrak{B}_2)_o\big) \cong \big(\mathfrak{A}_o \,\textcircled{\scriptsize u}\, (\mathfrak{B}_1 \oplus \mathfrak{B}_2)_o\big)\,.$$

Remark II.7.6

From the very beginning, we could have started with two coactions $C^*_{\mathrm{r}}(\mathcal{G}) \overset{\delta}{\curvearrowright} \mathfrak{A}$ and $C^*_{\mathrm{r}}(\mathcal{H}) \overset{\chi}{\curvearrowright} \mathfrak{B}$ by discrete (possibly, non-amenable) groups $\mathcal{G}$ and $\mathcal{H}$, instead of two actions of compact abelian groups, but the costruction of the algebraic twisted tensor product would turn out to reduce to our case. Indeed, as explained in Remark II.6.2, a bicharacter $u \in \mathsf{B}(\mathcal{G}, \mathcal{H})$ descends to $u_{\mathrm{ab}} \in \mathsf{B}(\mathcal{G}^{\mathrm{ab}}, \mathcal{H}^{\mathrm{ab}})$. Moreover, δ and χ induce coactions $C^*(\mathcal{G}^{\mathrm{ab}}) \overset{\delta_{\mathrm{ab}}}{\curvearrowright} \mathfrak{A}$ and $C^*(\mathcal{H}^{\mathrm{ab}}) \overset{\chi_{\mathrm{ab}}}{\curvearrowright} \mathfrak{B}$, or equivalently actions of the compact abelian groups $\widehat{\mathcal{G}^{\mathrm{ab}}} \overset{\alpha^{\delta_{\mathrm{ab}}}}{\curvearrowright} \mathfrak{A}$ and $\widehat{\mathcal{H}^{\mathrm{ab}}} \overset{\beta^{\chi_{\mathrm{ab}}}}{\curvearrowright} \mathfrak{B}$ (by Proposition II.4.3), where $\widehat{\mathcal{G}^{\mathrm{ab}}} \cong \mathsf{Hom}(\mathcal{G}, \mathbb{T})$ and $\widehat{\mathcal{H}^{\mathrm{ab}}} \cong \mathsf{Hom}(\mathcal{H}, \mathbb{T})$. Analogously to Proposition II.7.4, one gets

$$(\mathfrak{A}_{\mathcal{G}})_o \,\textcircled{\scriptsize u}\, (\mathfrak{B}_{\mathcal{H}})_o \cong (\mathfrak{A}_{\mathcal{G}^{\mathrm{ab}}})_{oo} \,\textcircled{\scriptsize u_{ab}}\, (\mathfrak{B}_{\mathcal{H}^{\mathrm{ab}}})_{oo}$$

where $\mathfrak{A}_{\mathcal{G}}$ and $\mathfrak{A}_{\mathcal{G}^{\mathrm{ab}}}$ are the same C^*-algebra $\mathfrak{A}$ graded respectively by $\mathcal{G}$ (where $(\mathfrak{A}_{\mathcal{G}})_g := \{a \in \mathfrak{A} \colon \delta(a) = a \otimes u_g\}$ for each $g \in \mathcal{G}$) and $\mathcal{G}^{\mathrm{ab}}$ (where $(\mathfrak{A}_{\mathcal{G}^{\mathrm{ab}}})_g := \{a \in \mathfrak{A} \colon \alpha_\sigma^{\delta_{\mathrm{ab}}}(a) = \sigma(g)a,\ \sigma \in \widehat{\mathcal{G}^{\mathrm{ab}}}\}$ for each $g \in \mathcal{G}^{\mathrm{ab}}$), and similarly for $\mathfrak{B}_{\mathcal{H}}$ and $\mathfrak{B}_{\mathcal{H}^{\mathrm{ab}}}$. For a reference, see Section 6.2 in [57], p. 32. As examples, if $\mathcal{G} = \mathfrak{S}_m$ $\mathcal{H} = \mathfrak{S}_n$ are the symmetric groups over m and n elements respectively $(m, n \geq 2)$, then $\mathcal{G}^{\mathrm{ab}} \cong \mathcal{H}^{\mathrm{ab}} \cong \mathbb{Z}_2$ and $u \in \mathsf{B}(\mathfrak{S}_m, \mathfrak{S}_n)$ descends to either the trivial or the Fermi bicharacter of $\mathbb{Z}_2$, i.e. either $u_{\mathrm{ab}} \equiv 1$ (in which case, there is no twist in the tensor product) or $u_{\mathrm{ab}}(x, y) = (-1)^{xy}$, $x, y \in \mathbb{Z}_2$ (a twisted case, thoroughly studied in [49], [17] and [31]). Analogously, if $\mathcal{G} = \mathcal{H} = Q_8 = \{1, \pm i, \pm j, \pm k\}$ is the (multiplicative) *quaternion group*, $\mathcal{G}^{\mathrm{ab}} = \mathcal{H}^{\mathrm{ab}} \cong K_4 = \mathbb{Z}_2 \times \mathbb{Z}_2$ (the Klein 4-group), so that $u \in \mathsf{B}(Q_8)$ descends to a bicharacter $u_{\mathrm{ab}} \in \mathsf{B}(K_4) \cong M_2(\mathbb{Z}_2)$. We shall extensively use a bicharacter $u \in \mathsf{B}(K_4)$ in the next chapter (see Section III.5) to deduce a new De Finetti-like theorem for infinite chains of twisted tensor products.

II.8. Twisted tensor product of representations

So far, we constructed algebraic twisted tensor products of two C^*-systems and proved some of their features. As for non-twisted tensor products of C^*-algebras, in order to get a C^*-completion we need to study representations on some (a priori, not necessarily complete) inner product spaces.

We then fix two C^*-systems $(\mathfrak{A}, G, \alpha)$, $(\mathfrak{B}, H, \beta)$ which inherit gradings on $\mathfrak{A}$ and $\mathfrak{B}$, and a bicharacter $u\colon \widehat{G} \times \widehat{H} \to \mathbb{T}$. The aim of the present section is to describe the twisted tensor product of representations of $\mathfrak{A}_o$ and $\mathfrak{B}_o$ (both necessarily having images lying in some algebras of *bounded* operators, thanks to Proposition II.5.2), thus obtaining representations of $\mathfrak{A}_o \textcircled{u} \mathfrak{B}_o$.

We start with a covariant representation $(\pi, \mathcal{H}_\pi, U)$, where $(\pi, \mathcal{H}_\pi)$ is a representation of $\mathfrak{A}_o$ and $G \ni g \mapsto U(g) \in \mathcal{U}(\mathcal{H})$ is a unitary representation of G which implements the action of G on $\mathfrak{A}_o$:

$$U(g)\pi(a)U(g)^* = \pi(\alpha_g(a)), \quad g \in G, \ a \in \mathfrak{A}_o .$$

At this stage, no continuity condition is assumed on the representation $U\colon G \to \mathcal{U}(\mathcal{H})$.[2]

If $(\rho, \mathcal{H}_\rho)$ is any representation of $\mathfrak{B}_o$, for any $a \in \mathfrak{A}_o$ and homogeneous $b \in \mathfrak{B}_o$, we set

$$\begin{aligned} &\big(\pi_U \textcircled{u} \rho\big)(a \odot b) := \pi(a)U(g_{\partial b}) \otimes \rho(b)\,, \\ &\text{with } g_{\partial b} \in G \text{ uniquely determined by} \\ &\sigma(g_{\partial b}) := \overline{u(\sigma, \partial b)}\,, \quad \sigma \in \widehat{G}\,. \end{aligned} \tag{II.13}$$

Proposition II.8.1
The map in (II.13) uniquely extends by linearity to the whole $\mathfrak{A}_o \odot \mathfrak{B}_o$, defining a representation (denoted by the same symbols) $\big(\pi_U \textcircled{u} \rho, \mathcal{H}_\pi \otimes \mathcal{H}_\rho\big)$ of the involutive algebra $\mathfrak{A}_o \textcircled{u} \mathfrak{B}_o$.

Proof.
Let $a, A \in \mathfrak{A}_o$ (a homogeneous) and $b, B \in \mathfrak{B}_o$ (b homogeneous). Since $\pi(a)U(g_{\partial b}) = \overline{\partial a(g_{\partial b})}U(g_{\partial b})\pi(a)$, we get

$$\begin{aligned} (\pi_U \textcircled{u} \rho)[(a \textcircled{u} b)^*] &= \overline{u(a,b)}\pi(a^*)U(g_{\partial b^*}) \otimes \rho(b^*) = (u(a,b)U(g_{\partial b})\pi(a) \otimes \rho(b))^* = \\ &= \Big(\overline{\partial a(g_{\partial b})}U(g_{\partial b})\pi(a) \otimes \rho(b)\Big)^* = (\pi(a)U(g_{\partial b}) \otimes \rho(b))^* = (\pi_U \textcircled{u} \rho)(a \textcircled{u} b)^* \end{aligned}$$

$$\begin{aligned} (\pi_U \textcircled{u} \rho)[(A \textcircled{u} b)(a \textcircled{u} B)] &= \overline{u(a,b)}\pi(Aa)U(g_{\partial(bB)}) \otimes \rho(bB) = \\ &= \overline{u(a,b)}\pi(A)\pi(a)U(g_{\partial b})U(g_{\partial B}) \otimes \rho(b)\rho(B) = \\ &= \partial a(g_{\partial b})\pi(A)\pi(a)U(g_{\partial b})U(g_{\partial B}) \otimes \rho(b)\rho(B) = \\ &= \pi(A)U(g_{\partial b})\pi(a)U(g_{\partial B}) \otimes \rho(b)\rho(B) = (\pi_U \textcircled{u} \rho)(A \textcircled{u} b)(\pi_U \textcircled{u} \rho)(a \textcircled{u} B) \end{aligned}$$

We are left to check that (II.13) is a well-defined map between $\mathfrak{A}_o \odot \mathfrak{B}_o$ (isomorphic to $\mathfrak{A}_o \textcircled{u} \mathfrak{B}_o$ and $\mathfrak{A}_o \otimes \mathfrak{B}_o$ as linear spaces) and $\mathcal{B}\big(\mathcal{H}_\pi \otimes \mathcal{H}_\rho\big)$. Since $\mathfrak{A}_o \odot \mathfrak{B}_o = \dotplus_{\tau \in \widehat{H}} \mathfrak{A}_o \odot \mathfrak{B}_\tau$, it is enough to prove the assertion on each direct summand. But, for $\tau \in \widehat{H}$, all maps

$$\mathfrak{A}_o \odot \mathfrak{B}_\tau \ni a \odot b \mapsto (\pi(a) \otimes \rho(b))(U(g_\tau) \otimes I_{\mathcal{H}_\rho}) \in \mathcal{B}\big(\mathcal{H}_\pi \otimes \mathcal{H}_\rho\big)$$

are manifestly well defined. $\square$

[2]Here, it would be enough to assume covariance only for the subgroup generated by $\{g_\tau : \tau \in \mathrm{spec}_o(\beta)\} \subset G$ where g_τ is determined in the forthcoming (II.13).

Equally well, one can start with a representation $(\pi, \mathcal{H}_\pi)$ of $\mathfrak{A}_o$ and a covariant representation $(\rho, \mathcal{H}_\rho, V)$ of $\mathfrak{B}_o$. For $a \in \mathfrak{A}_o$ homogeneous and any $b \in \mathfrak{B}_o$, define

$$\begin{aligned} &\big(\pi \circledast_V \rho\big)(a \odot b) := \pi(a) \otimes V({}_{\partial a}h)\rho(b)\,,\\ &\text{with } {}_{\partial a}h \in H \text{ uniquely determined by}\\ &\chi_{{}_{\partial a}h}(\tau) := u(\partial a, \tau)\,, \quad \tau \in \widehat{H}\,. \end{aligned} \tag{II.14}$$

It is easy to check that also (II.14) defines a representation $\big(\pi \circledast_V \rho, \mathcal{H}_\pi \otimes \mathcal{H}_\rho\big)$ of the involutive algebra $\mathfrak{A}_o \circledast \mathfrak{B}_o$.

Remark II.8.2
A unitary representation $G \ni g \mapsto U_g \in \mathcal{H}(\mathcal{U})$ induces a $\widehat{G}$-grading on $\mathcal{H}$ provided by the (closed) *spectral subspaces*

$$\mathcal{H}_\sigma := \{\xi \in \mathcal{H} \colon U_g\xi = \sigma(g)\xi, g \in G\} = \bigcap_{g\in G} \ker(U_g - \sigma(g)I)\,.$$

Observe that $\mathcal{H}_\iota = \mathcal{H}^G = \{\xi \in \mathcal{H} \colon U_g\xi = \xi,\, g \in G\} = \bigcap_{g\in G} \ker(U_g - I)$, $\mathcal{H}_\sigma \perp \mathcal{H}_\tau$ for every pair of distinct $\sigma, \tau \in \widehat{G}$ and $\mathcal{H}_o := \operatorname{span}_{\mathbb{C}}\{\mathcal{H}_\sigma \colon \sigma \in \widehat{G}\}$ is a dense pre-Hilbert subspace of $\mathcal{H}$. If the representation U is strongly continuous, the $\widehat{G}$-grading on $\mathcal{H}$ induces a $\widehat{G}$-grading on the compact operators $\mathcal{K}(\mathcal{H})$ by $(\mathrm{ad}_{U_g})_{g\in G}$ satisfying

$$\partial(A\xi) = \partial(A)\partial(\xi)$$

for every homogeneous compact operator $A \in \mathcal{K}(\mathcal{H})$ and homogeneous vector $\xi \in \mathcal{H}$. In general, the adjoint action of a strongly continuous unitary representation is not pointwise norm-continuous on the whole $\mathcal{B}(\mathcal{H})$.
Lastly, if $(\mathcal{H}, (U_g)_{g\in G})$ and $(\mathcal{K}, (V_h)_{h\in H})$ are two Hilbert spaces, respectively graded by $\widehat{G}$ and $\widehat{H}$, then their tensor product Hilbert space $\mathcal{H} \otimes \mathcal{K}$ inherits a natural $(\widehat{G} \times \widehat{H})$-grading given by the unitary representation $(U_g \otimes V_h)_{(g,h)\in G\times H}$ of $G \times H$.

II.9. Product states and their GNS representations

In order to exhibit a genuine twisted C^*-tensor product, we need to complete $\mathfrak{A}_o \circledast \mathfrak{B}_o$ w.r.t. some C^*-norm, which in general is not unique, as in the non-twisted case. As in [17], [31], the starting point is a detailed investigation of product states.
Given two linear spaces X and Y and linear functionals $f \in X'$ and $g \in Y'$, $f \odot g$ given by

$$(f \odot g)\Big(\sum_k a_k \odot b_k\Big) := \sum_k f(a_k)g(b_k)\,, \quad \sum_k a_k \odot b_k \in X \odot Y$$

is the *product functional* of f and g, provided it is well defined. Suppose now $X = \mathfrak{A}$, $Y = \mathfrak{B}$ are two C^*-algebras and $f = \omega$, $g = \varphi$ two positive functionals, then belonging to the corresponding topological duals $\mathfrak{A}^*$ and $\mathfrak{B}^*$. The product functional of ω and φ will be denoted by $\omega \times \varphi$ and $\psi_{\omega,\varphi}$ when it is considered on the involutive algebras $\mathfrak{A}_o \circledast \mathfrak{B}_o$ and $\mathfrak{A}_o \otimes \mathfrak{B}_o$, respectively.
We immediately deduce that $\omega \odot \varphi$ is certainly well defined because it coincides with the product functional $\psi_{\omega,\varphi}$. Such a product functional is also positive when considered even on the whole involutive algebra $\mathfrak{A} \otimes \mathfrak{B}$. Instead, on $\mathfrak{A}_o \circledast \mathfrak{B}_o$, the product functional $\omega \times \varphi$ is always well defined, but possibly not positive. We directly show that the positivity of the product functional $\omega \times \varphi$ is guaranteed whenever either ω or φ is invariant under the group action. [3]

[3] Notice that this result can also be deduced from the discussion in Section II.8 by constructing the twisted product of the GNS representations according to (II.13) or (II.14).

Proposition II.9.1
For the product functional $\omega\times\varphi$, $\omega\in\mathcal{S}(\mathfrak{A})$ and $\varphi\in\mathcal{S}(\mathfrak{B})$, suppose that at least one of them is invariant (i.e. either $\omega\in\mathcal{S}_G(\mathfrak{A})$ or $\varphi\in\mathcal{S}_H(\mathfrak{B})$). Then, $\omega\times\varphi$ is a state on $\mathfrak{A}_o\circledast\mathfrak{B}_o$, in which case

$$|(\omega\times\varphi)(x)|\leq(\omega\times\varphi)(x^*x)^{1/2}\,,\quad x\in\mathfrak{A}_o\circledast\mathfrak{B}_o\,.$$

Proof.
First of all, observe that for every $a=\sum_\sigma a_\sigma\in\mathfrak{A}_o$, $b=\sum_\tau b_\tau\in\mathfrak{B}_o$, $A=\sum_s A_s\in\mathfrak{A}_o$, $B\in\mathfrak{B}_o$

$$\begin{aligned}(a\odot b)^*(A\odot B)&=\sum_{\sigma,s,\tau}\overline{u(\sigma s^{-1},\tau)}(a_\sigma\odot b_\tau)^\dagger\cdot(A_s\odot B)\\&=\sum_\tau\Big(\sum_\sigma u(\sigma,\tau)a_\sigma\odot b_\tau\Big)^\dagger\cdot\Big(\sum_s u(s,\tau)A_s\odot B\Big)\\&=\sum_\tau\big(\alpha^{-1}_{g_\tau}(a)\odot b_\tau\big)^\dagger\cdot\big(\alpha^{-1}_{g_\tau}(A)\odot B\big)\end{aligned}$$

where we have used the fact that, for each fixed $\sigma\in\widehat{G},\tau\in\widehat{H}$,

$$u(\sigma,\tau)a_\sigma=\overline{\sigma(g_\tau)}a_\sigma=\alpha^{-1}_{g_\tau}(a_\sigma)\,.$$

At this point, to achieve the positivity of $\omega\times\varphi$, we shall show that

$$(\omega\times\varphi)(x^*x)=\psi_{\omega,\varphi}(x^\dagger\cdot x)\geq 0$$

for every $x\in\mathfrak{A}_o\circledast\mathfrak{B}_o$, whenever either $\omega\in\mathcal{S}_G(\mathfrak{A})$ or $\varphi\in\mathcal{S}_H(\mathfrak{B})$. By Proposition II.7.5, it is enough to consider the former case.
By G-invariance of ω,

$$(\omega\times\varphi)\big((a\odot b)^*(A\odot B)\big)=\sum_\tau\psi_{\omega,\varphi}(\big(\alpha^{-1}_{g_\tau}(a)\odot b_\tau\big)^\dagger\cdot\big(\alpha^{-1}_{g_\tau}(A)\odot B\big))$$

$$=\sum_\tau\psi_{\omega,\varphi}((a\odot b_\tau)^\dagger\cdot(A\odot B))=\psi_{\omega,\varphi}((a\odot b)^\dagger\cdot(A\odot B))\,,$$

and thus for every finite linear combination $x:=\sum_{k=1}^n a^{(k)}\odot b^{(k)}$, $a^{(k)}\in\mathfrak{A}_o$, $b^{(k)}\in\mathfrak{B}_o$, we get

$$\begin{aligned}(\omega\times\varphi)(x^*x)=&\sum_{k,l=1}^n\psi_{\omega,\varphi}\Big(\big(a^{(k)}\odot b^{(k)}\big)^\dagger\cdot\big(a^{(l)}\odot b^{(l)}\big)\Big)\\=&\psi_{\omega,\varphi}\big(x^\dagger\cdot x\big)\geq 0\,.\end{aligned}\tag{II.15}$$

The proof follows since the inequality in the statement is nothing but the Cauchy-Bunyakovsky-Schwarz inequality. □

Remark II.9.2
An equality in the proof of the assertion above in the special case $G=H=\mathbb{Z}_2$, given in [17] (p. 18), may not hold. Precisely, let $\omega\in\mathcal{S}_{\mathbb{Z}_2}(\mathfrak{A})$, $\varphi\in\mathcal{S}(\mathfrak{B})$. Then, in general, if $x,y\in\mathfrak{A}\circledF\mathfrak{B}$,

$$(\omega\times\varphi)((yx)^*(yx))=\psi_{\omega,\varphi}((yx)^\dagger\cdot(yx))\neq\psi_{\omega,\varphi}((y\cdot x)^\dagger\cdot(y\cdot x))\,.$$

For instance, let $\mathfrak{A}:=\mathcal{C}(\mathbb{T})=C^*(u\colon u\text{ unitary})$, $\mathfrak{B}:=\mathcal{C}(\mathbb{T})=C^*(v\colon v\text{ unitary})$, both equipped with the involutive automorphism

$$\vartheta\colon f\mapsto[z\mapsto f(-z)]$$

Then, $\mathfrak{A} = \underbrace{[z^{2k} \colon k \in \mathbb{Z}]}_{\mathfrak{A}_0} \oplus \underbrace{[z^{2k+1} \colon k \in \mathbb{Z}]}_{\mathfrak{A}_1}$. Let $\omega \in \mathcal{S}_{\mathbb{Z}_2}(\mathfrak{A})$ be the unique state on $\mathfrak{A}$ s.t. $\omega(z^{2k}) = 1$, $\omega(z^{2k+1}) = 0$, $k \in \mathbb{Z}$. If $x = y = (1+u) \odot (1+v)$, we have

$$\psi_{\omega,\omega}((xx)^{\dagger} \cdot (xx)) \neq \psi_{\omega,\omega}((x \cdot x)^{\dagger} \cdot (x \cdot x)) .$$

Indeed,

$$(xx)^{\dagger} \cdot (xx) = \sum_{j,h,j',h'=0}^{1} (-1)^{jh+j'h'} (u^{h'-h}|1+u|^2) \odot (v^{j'-j}|1+v|^2)$$

and thus

$$\psi_{\omega,\omega}((xx)^{\dagger} \cdot (xx)) = \sum_{j,h,j',h'=0}^{1} (-1)^{jh+j'h'} \omega(u^{h'-h}|1+u|^2) \omega(v^{j'-j}|1+v|^2)$$

Now, $|1+u|^2 = 2 + u + \overline{u}$, thus $\omega(|1+u|^2) = \omega(u|1+u|^2) = \omega(\overline{u}|1+u|^2) = 2$. It follows that

$$\psi_{\omega,\omega}((xx)^{\dagger} \cdot (xx)) = 4 \sum_{j,h,j',h'=0}^{1} (-1)^{jh+j'h'} = 4(2^4 - 6) = 40$$

On the other hand,

$$\psi_{\omega,\omega}((x \cdot x)^{\dagger} \cdot (x \cdot x)) = 4 \sum_{j,h,j',h'=0}^{1} 1 = 4 \cdot 2^4 = 64 .$$

In conclusion, $\psi_{\omega,\omega}((xx)^{\dagger} \cdot (xx)) \neq \psi_{\omega,\omega}((x \cdot x)^{\dagger} \cdot (x \cdot x))$.

Notice that, by looking at (II.15), we have incidentally proved the following crucial

Proposition II.9.3
The set of the invariant product states $\{\omega \times \varphi \colon \omega \in \mathcal{S}_G(\mathfrak{A}),\ \varphi \in \mathcal{S}_H(\mathfrak{B})\}$ separates the points of $\mathfrak{A}_o \circledast \mathfrak{B}_o$.

Proof.
We start by noticing that, by point (iii) of Theorem IV.4.9 in [104] (p. 208), if $(\pi, \mathcal{H})$ and $(\sigma, \mathcal{K})$ are two faithful representations of $\mathfrak{A}$ and $\mathfrak{B}$ respectively, then the algebraic tensor product map $\pi \otimes \sigma$ extends to a faithful representation of $\mathfrak{A} \otimes_{\min} \mathfrak{B}$ on the Hilbert space $\mathcal{H} \otimes \mathcal{K}$. It follows that, given a non-zero $x \in \mathfrak{A} \otimes_{\min} \mathfrak{B}$, there must exist a normalized elementary tensor vector $\xi \otimes \eta \in \mathcal{H} \otimes \mathcal{K}$ (depending on x) such that

$$\big\|(\pi \otimes \sigma)(x)(\xi \otimes \eta)\big\|^2 = \big\langle (\pi \otimes \sigma)(x^{\dagger} \cdot x)\xi \otimes \eta, \xi \otimes \eta \big\rangle > 0 .$$

In other words, for a fixed $\mathfrak{A} \otimes_{\min} \mathfrak{B} \ni x \neq 0$, there exists a product state $\omega_{\xi\otimes\eta} = \psi_{\omega_\xi,\omega_\eta}$ such that $\omega_{\xi\otimes\eta}(x^{\dagger} \cdot x) > 0$. In addition, since the expectations $E_G : \mathfrak{A} \to \mathfrak{A}^G$ and $E_H : \mathfrak{A} \to \mathfrak{A}^H$ are faithful, $E_G \otimes E_H : \mathfrak{A} \otimes_{\min} \mathfrak{B} \to \mathfrak{A}^G \otimes_{\min} \mathfrak{B}^H$ is also faithful (see Corollary at the end of the Appendix in [4], p. 434).
Collecting all things together, for such a fixed $x \in \mathfrak{A}_o \circledast \mathfrak{B}_o$ (the latter being linearly isomorphic to $\mathfrak{A}_o \otimes \mathfrak{B}_o$),

$$\Big(\big(\omega_\xi \circ E_G\big) \times \big(\omega_\eta \circ E_H\big)\Big)(x^* x) = \psi_{\omega_\xi \circ E_G, \omega_\eta \circ E_H}(x^{\dagger} \cdot x) = \psi_{\omega_\xi,\omega_\eta}\big((E_G \otimes E_H)(x^{\dagger} \cdot x)\big) > 0 . \quad \square$$

If ω and φ are invariant states on $\mathfrak{A}_o$ and $\mathfrak{B}_o$ (which are in 1-1 bijection with invariant states on $\mathfrak{A}$ and $\mathfrak{B}$, respectively), we can look at the corresponding GNS covariant representations $(\pi_\omega, \mathcal{H}_\omega, U_\omega, \xi_\omega)$ and $(\pi_\varphi, \mathcal{H}_\varphi, U_\varphi, \xi_\varphi)$, and build the representations (II.13) and (II.14). In addition, both of them are equipped with the cyclic vector $\xi_\omega \otimes \xi_\varphi$. It is now straightforward to check that both the representations realize a GNS representation of the product state $\omega \times \varphi$. We summarize the property of the GNS representation of product states in the following

Theorem II.9.4
Let $(\pi_\omega, \mathcal{H}_\omega, V_\omega, \xi_\omega)$ and $(\pi_\varphi, \mathcal{H}_\varphi, V_\varphi, \xi_\varphi)$ the GNS representations of the invariant states $\omega \in \mathcal{S}_G(\mathfrak{A})$ and $\varphi \in \mathcal{S}_H(\mathfrak{B})$, respectively.
The GNS representation $(\mathcal{H}_{\omega\times\varphi}, \pi_{\omega\times\varphi}, \xi_{\omega\times\varphi})$ of $\mathfrak{A}_o \textcircled{w} \mathfrak{B}_o$ relative to the product state $\omega \times \varphi$ is given (up to unitary equivalence) by $(\mathcal{H}_\omega \otimes \mathcal{H}_\varphi, \pi_\omega \textcircled{w} \pi_\varphi, \xi_\omega \otimes \xi_\varphi)$ with $\pi_\omega \textcircled{w} \pi_\varphi$ one of the representations in (II.13), (II.14).

II.10. On representations of the twisted tensor product

We have already seen that the class of representations of the involutive algebra $\mathfrak{A}_o \textcircled{w} \mathfrak{B}_o$ is rich enough, as it contains the GNS representations of product states consisting of invariant ones (cf. Theorem II.9.4). Furthermore, this class of states is separating for $\mathfrak{A}_o \textcircled{w} \mathfrak{B}_o$ (cf. Proposition II.9.3). On the other hand, on a general *-algebra, the GNS construction induced by positive functionals may well produce unbounded operators defined on a common core (see *e.g.* [17], Theorem 3.2, p. 7).- We start with showing that this is not the case for $\mathfrak{A}_o \textcircled{w} \mathfrak{B}_o$, with a technique similar to the one used in the proof of Proposition II.5.2 for the single marginal algebras $\mathfrak{A}_o$ and $\mathfrak{B}_o$.
Recall that $\sum(\mathfrak{A}_o \textcircled{w} \mathfrak{B}_o)^2$ is the cone generated by the positive elements z^*z of $\mathfrak{A}_o \textcircled{w} \mathfrak{B}_o$:

$$\sum(\mathfrak{A}_o \textcircled{w} \mathfrak{B}_o)^2 := \left\{ \sum_{i=1}^{n} z_i^* z_i \colon z_i \in \mathfrak{A}_o \textcircled{w} \mathfrak{B}_o, n \geq 1 \right\} \subset \mathfrak{A}_o \textcircled{w} \mathfrak{B}_o \,.$$

Evidently, $\sum(\mathfrak{A}_o \textcircled{w} \mathfrak{B}_o)^2$ is convex (or, equivalently, closed under addition), hence it induces a partial order on $\mathfrak{A}_o \textcircled{w} \mathfrak{B}_o$, as usual: for $s, t \in \mathfrak{A}_o \textcircled{w} \mathfrak{B}_o$, we write $s \leq_{\textcircled{w}} t$ if $t - s \in \sum(\mathfrak{A}_o \textcircled{w} \mathfrak{B}_o)^2$. Moreover, $\sum(\mathfrak{A}_o \textcircled{w} \mathfrak{B}_o)^2$ is a *quadratic module* of $\mathfrak{A}_o \textcircled{w} \mathfrak{B}_o$, that is $\mathbb{1}_\mathfrak{A} \textcircled{w} \mathbb{1}_\mathfrak{B} \in \sum(\mathfrak{A}_o \textcircled{w} \mathfrak{B}_o)^2$ and $x^* s x \in \sum(\mathfrak{A}_o \textcircled{w} \mathfrak{B}_o)^2$ for every $s \in \sum(\mathfrak{A}_o \textcircled{w} \mathfrak{B}_o)^2$ and $x \in \mathfrak{A}_o \textcircled{w} \mathfrak{B}_o$.
The following result shows a crucial property of this quadratic module, analogue to the one in the usual non-twisted setting (see [24], Lemma 2.1 (ii), p. 7).

Proposition II.10.1 (Lance-Effros inequality for twisted tensor products)
The quadratic module $\sum(\mathfrak{A}_o \textcircled{w} \mathfrak{B}_o)^2$ is Archimedean, that is for each $y \in \mathfrak{A}_o \textcircled{w} \mathfrak{B}_o$, there exists a positive constant $C_y > 0$ (only depending on y) such that $y^* y \leq_{\textcircled{w}} C_y(\mathbb{1}_\mathfrak{A} \textcircled{w} \mathbb{1}_\mathfrak{B})$.

Proof.
Let $y := \sum_{i=1}^{n} a^{(i)} \textcircled{w} b^{(i)} \in \mathfrak{A}_o \textcircled{w} \mathfrak{B}_o$, with $a^{(i)} = \sum_{\sigma \in S_i} a_\sigma^{(i)} \in \mathfrak{A}_o, b^{(i)} = \sum_{\tau \in T_i} b_\tau^{(i)} \in \mathfrak{B}_o$ (where $S_i \subset \widehat{G}$, $T_i \subset \widehat{H}$ are the finite supports of $a^{(i)}$ and $b^{(i)}$ respectively). Then, once set $S := \bigcup_{i=1}^{n} S_i$ and

$T := \bigcup_{i=1}^{n} T_i$, we can write

$$y^*y = \sum_{i,j=1}^{n} \sum_{\tau \in T_i} \alpha_{g_\tau}^{-1}(a^{(i)*}a^{(j)}) \textcircled{w} b_\tau^{(i)*} b^{(j)} = \sum_{\tau,t \in T} \sum_{i,j=1}^{n} \alpha_{g_\tau}^{-1}(a^{(i)*}a^{(j)}) \textcircled{w} b_\tau^{(i)*} b_t^{(j)}$$

where we used Pontryagin duality to locate the unique element $g_\tau \in G$ such that $u(\cdot,\tau) = \chi_{g_\tau} \in \widehat{\widehat{G}}$.
Our aim now is to define a positive $Y \in \mathfrak{A}_o \textcircled{w} \mathfrak{B}_o$ such that

$$y^*y \leq_{\textcircled{w}} y^*y + Y \leq_{\textcircled{w}} C_y(\mathbb{1}_{\mathfrak{A}} \textcircled{w} \mathbb{1}_{\mathfrak{B}}),$$

for a suitable $C_y > 0$. To accomplish that, we totally order the finite sets $S \subset \widehat{G}, T \subset \widehat{H}$ by binary relations $\preceq_S$ and $\preceq_T$, respectively. For each of the $\binom{|T|}{2}$ pairs $(\tau, t) \in T \times T$ s.t. $\tau \prec_T t$, let $Z_{\tau,t} := \sum_{i=1}^{n} a^{(i)} \textcircled{w} (b_\tau^{(i)} - b_t^{(i)})$. Then,

$$Z_{\tau,t}^* Z_{\tau,t} = \sum_{i,j=1}^{n} \alpha_{g_\tau}^{-1}(a^{(i)*}a^{(j)}) \textcircled{w} b_\tau^{(i)*} b_\tau^{(j)} - \sum_{i,j=1}^{n} \alpha_{g_\tau}^{-1}(a^{(i)*}a^{(j)}) \textcircled{w} b_\tau^{(i)*} b_t^{(j)}$$
$$- \sum_{i,j=1}^{n} \alpha_{g_t}^{-1}(a^{(i)*}a^{(j)}) \textcircled{w} b_t^{(i)*} b_\tau^{(j)} + \sum_{i,j=1}^{n} \alpha_{g_t}^{-1}(a^{(i)*}a^{(j)}) \textcircled{w} b_t^{(i)*} b_t^{(j)},$$

whence $y^*y + \sum_{\tau \prec_T t} Z_{\tau,t}^* Z_{\tau,t} = |T| \sum_{\tau \in T} \sum_{i,j=1}^{n} \alpha_{g_\tau}^{-1}(a^{(i)*}a^{(j)}) \textcircled{w} b_\tau^{(i)*} b_\tau^{(j)}$.

Similarly, once fixed $\tau \in T$, for each of the $\binom{|S|}{2}$ pairs $(\sigma, s) \in S \times S$ such that $\sigma \prec_S s$, let $W_{\sigma,s}^{(\tau)} := \sum_{i=1}^{n} (a_\sigma^{(i)} - a_s^{(i)}) \textcircled{w} b_\tau^{(i)}$. Then,

$$W_{\sigma,s}^{(\tau)*} W_{\sigma,s}^{(\tau)} = \sum_{i,j=1}^{n} \alpha_{g_\tau}^{-1}\left(a_\sigma^{(i)*}a_\sigma^{(j)} + a_s^{(i)*}a_s^{(j)} - a_\sigma^{(i)*}a_s^{(j)} - a_s^{(i)*}a_\sigma^{(j)}\right) \textcircled{w} b_\tau^{(i)*} b_\tau^{(j)}$$

and $\underbrace{y^*y + \sum_{\tau \prec_T t} Z_{\tau,t}^* Z_{\tau,t} + |T| \sum_{\tau \in T} \sum_{\sigma \prec_S s} W_{\sigma,s}^{(\tau)*} W_{\sigma,s}^{(\tau)}}_{=:Y} = |S||T| \sum_{\substack{\sigma \in S \\ \tau \in T}} \mathcal{Y}_{\sigma,\tau}^\dagger \cdot \mathcal{Y}_{\sigma,\tau}$ where $\mathcal{Y}_{\sigma,\tau} := \sum_{i=1}^{n} a_\sigma^{(i)} \textcircled{w} b_\tau^{(i)}$.

Therefore, $y^*y + Y$ is positive not only as an element of $\mathfrak{A}_o \textcircled{w} \mathfrak{B}_o$, but also as one of $\mathfrak{A} \otimes \mathfrak{B}$. This fact allows us to exploit the polarization identity for every pair $(\sigma, \tau) \in S \times T$ to write

$$\mathcal{Y}_{\sigma,\tau}^\dagger \cdot \mathcal{Y}_{\sigma,\tau} = \frac{1}{4} \sum_{i,j=1}^{n} \underbrace{(a_\sigma^{(i)*}a_\sigma^{(j)} + a_\sigma^{(j)^*}a_\sigma^{(i)})}_{=:A_{ij} \in \mathfrak{A}_{sa}^G} \textcircled{w} \underbrace{(b_\tau^{(i)*}b_\tau^{(j)} + b_\tau^{(j)*}b_\tau^{(i)})}_{=:B_{ij} \in \mathfrak{B}_{sa}^H} +$$
$$+ \underbrace{i(a_\sigma^{(j)*}a_\sigma^{(i)} - a_\sigma^{(i)^*}a_\sigma^{(j)})}_{=:A'_{ij} \in \mathfrak{A}_{sa}^G} \textcircled{w} \underbrace{i(b_\tau^{(i)*}b_\tau^{(j)} - b_\tau^{(j)*}b_\tau^{(i)})}_{=:B'_{ij} \in \mathfrak{B}_{sa}^H} =$$
$$= \frac{1}{4} \sum_{i,j=1}^{n} (A_{ij,+} - A_{ij,-}) \textcircled{w} B_{ij} + (A'_{ij,+} - A'_{ij,-}) \textcircled{w} B'_{ij},$$

where $X_\pm = \dfrac{\|X\| \pm X}{2} \in \mathfrak{A}^G$ (for $X = A_{ij}, A'_{ij}$) are positive elements in $\mathfrak{A}$.[4] Here comes the crucial point: since $A_{ij}, \|A_{ij,\pm}\| - A_{ij,\pm} \in \mathfrak{A}^G$ are positive in $\mathfrak{A}$, they both admit square roots $\sqrt{A_{ij,\pm}}, \sqrt{\|A_{ij,\pm}\| - A_{ij,\pm}} \in C^*(A_{ij,\pm}) \subset \mathfrak{A}^G$. Analogously, since B_{ij} is selfadjoint, $\|B_{ij}\| - B_{ij} \in \mathfrak{B}^H$ is positive in $\mathfrak{B}$ and $\sqrt{\|B_{ij}\| - B_{ij}} \in \mathfrak{B}^H$. On the one hand, it follows that

$$
\begin{aligned}
&(\|A_{ij,+}\| - A_{ij,+}) \circledast B_{ij} = \\
&\qquad = \left(\sqrt{\|A_{ij,+}\| - A_{ij,+}} \circledast \sqrt{B_{ij}}\right)^* \left(\sqrt{\|A_{ij,+}\| - A_{ij,+}} \circledast \sqrt{B_{ij}}\right) \geq_{\circledast} 0
\end{aligned}
$$

and similarly $A_{ij,+} \circledast (\|B_{ij}\| - B_{ij}) \geq_{\circledast} 0$, whence

$$
\begin{aligned}
A_{ij,+} \circledast B_{ij} \leq_{\circledast} & \|A_{ij,+}\| \|B_{ij}\| (\mathbb{1}_{\mathfrak{A}} \circledast \mathbb{1}_{\mathfrak{B}}) \leq_{\circledast} \|A_{ij}\| \|B_{ij}\| (\mathbb{1}_{\mathfrak{A}} \circledast \mathbb{1}_{\mathfrak{B}}) \\
& \leq_{\circledast} 4 \|a_\sigma^{(i)}\| \|a_\sigma^{(j)}\| \|b_\tau^{(i)}\| \|b_\tau^{(j)}\| (\mathbb{1}_{\mathfrak{A}} \circledast \mathbb{1}_{\mathfrak{B}}) \,.
\end{aligned}
$$

On the other hand,

$$
-A_{ij,-} \circledast B_{ij} = -(\sqrt{A_{ij,-}} \circledast \sqrt{B_{ij}})^* (\sqrt{A_{ij,-}} \circledast \sqrt{B_{ij}}) \leq_{\circledast} 0 \,.
$$

In the very same way, $A'_{ij,+} \circledast B'_{ij} \leq_{\circledast} 4 \|a_\sigma^{(i)}\| \|a_\sigma^{(j)}\| \|b_\tau^{(i)}\| \|b_\tau^{(j)}\| (\mathbb{1}_{\mathfrak{A}} \circledast \mathbb{1}_{\mathfrak{B}})$ and $-A'_{ij,-} \circledast B'_{ij} \leq_{\circledast} 0$. All things considered,

$$
\begin{aligned}
\mathcal{Y}^\dagger_{\sigma,\tau} \cdot \mathcal{Y}_{\sigma,\tau} \leq_{\circledast} & \frac{1}{4} \sum_{i,j=1}^n \|A_{ij}\| \|B_{ij}\| + \|A'_{ij}\| \|B'_{ij}\| \\
\leq_{\circledast} & 2 \left(\sum_{i=1}^n \|a_\sigma^{(i)}\| \|b_\tau^{(i)}\| \right)^2 (\mathbb{1}_{\mathfrak{A}} \circledast \mathbb{1}_{\mathfrak{B}})
\end{aligned}
$$

and $y^* y \leq_{\circledast} y^* y + Y \leq_{\circledast} \underbrace{2|S||T| \sum_{\substack{\sigma \in S \\ \tau \in T}} \left(\sum_{i=1}^n \|a_\sigma^{(i)}\| \|b_\tau^{(i)}\| \right)^2}_{=: C_y} (\mathbb{1}_{\mathfrak{A}} \circledast \mathbb{1}_{\mathfrak{B}})$. We conclude that the quadratic module $\sum (\mathfrak{A}_o \circledast \mathfrak{B}_o)^2$ is Archimedean. □

Thanks to the previous theorem the involutive algebra $\mathfrak{A}_o \circledast \mathfrak{B}_o$ is (algebraically) bounded, as defined in Section II.3, and all its representations consist of bounded operators on Hilbert spaces (see Lemma II.3.1).

Whenever we have a C^*-norm $\| \ \|_\gamma$ on the twisted tensor product $\mathfrak{A}_o \circledast \mathfrak{B}_o$, simply denoted by γ, we write (with an abuse of notation motivated by the forthcoming Proposition II.11.5) $\mathfrak{A} \circledast_\gamma \mathfrak{B}$ for its completion w.r.t. γ.
Since the actions of G and H on $\mathfrak{A}$ and $\mathfrak{B}$ easily extend by linearity on the twisted tensor product to an action of $G \times H$ defined on generators by

$$
(\alpha_g \times \beta_h)(a \odot b) := \alpha_g(a) \odot \beta_h(b) \,, \quad a \in \mathfrak{A}_o, \ b \in \mathfrak{B}_o, \ g \in G, \ h \in H \,,
$$

a crucial question is whether this action extends to the completion $\mathfrak{A} \circledast_\gamma \mathfrak{B}$ w.r.t. a fixed norm γ. Another question concerns the cross property of a given norm.

[4] Notice that $A_{ij}, A'_{ij}, B_{ij}, B'_{ij}$ clearly depend on the choice of the pair (σ, τ) but we do not explicit this dependence here, to avoid making the notation heavier.

Definition II.10.2
Let γ be a C^*-norm on the twisted tensor product $\mathfrak{A}_o \textcircled{w} \mathfrak{B}_o$.

(a) The norm γ is said to be *compatible* if all $\alpha_g \times \beta_h$, $g \in G$ and $h \in H$, extend to contractive maps, and thus to *-automorphisms, on the completion $\mathfrak{A} \textcircled{w}_\gamma \mathfrak{B}$;

(b_0) γ is said to be *sub-cross* if

$$\|a \textcircled{w} b\|_\gamma \leq \|a\|_{\mathfrak{A}} \|b\|_{\mathfrak{B}}\,, \quad a \in \mathfrak{A}_o,\ b \in \mathfrak{B}_o\,;$$

(b) γ is said to be *cross* if

$$\|a \textcircled{w} b\|_\gamma = \|a\|_{\mathfrak{A}} \|b\|_{\mathfrak{B}}\,,$$

whenever either $a \in \mathfrak{A}_o$ or $b \in \mathfrak{B}_o$ is homogeneous.

Remark II.10.3
Our definition of cross property of a C^*-norm follows the one of the non-twisted setting, but keeps track of the grading structure of the two factors. It looks more restrictive than the one provided in [19] (Definition 4.2, p. 12), yet all the proofs in there still perfectly work with our definition. Our strengthening is mainly aimed at guaranteeing the isometric embedding of each marginal C^*-algebra into the completion w.r.t. a cross C^*-norm, as Proposition II.11.5 will show.

For a compatible norm γ, the action $\alpha \times \beta$ extends to an action of the compact group $G \times H$ on $\mathfrak{A} \textcircled{w}_\gamma \mathfrak{B}$ denoted by $\alpha \textcircled{w}_\gamma \beta$. By following the lines of Proposition II.5.3, we show that such an action is actually pointwise norm-continuous.

Proposition II.10.4
Let γ be a compatible C^*-norm on $\mathfrak{A}_o \textcircled{w} \mathfrak{B}_o$. Then, the action $\alpha \textcircled{w}_\gamma \beta$ of $G \times H$ on $\mathfrak{A} \textcircled{w}_\gamma \mathfrak{B}$ is pointwise norm-continuous, and thus yielding a C^*-system $\big(\mathfrak{A} \textcircled{w}_\gamma \mathfrak{B}, G \times H, \alpha \textcircled{w}_\gamma \beta\big)$.

Proof.
The same 2ε-argument of Proposition II.5.3 leads to the assertion. Indeed, fixing $\varepsilon > 0$ and $x \in \mathfrak{A} \textcircled{w}_\gamma \mathfrak{B}$, we find

$$x_\varepsilon = \sum_{j=1}^n a_j \odot b_j\,, \quad \text{with the } a_j\text{'s and } b_j\text{'s homogeneous}\,,$$

such that $\|x - x_\varepsilon\|_\gamma \leq \varepsilon$. We then get

$$\|(\alpha \textcircled{w}_\gamma \beta)_{(g,h)}(x) - x\|_\gamma \leq 2\varepsilon + \sum_{j=1}^n |\partial a_j(g) \partial b_j(h) - 1| \|a_j \odot b_j\|_\gamma\,.$$

Taking the lim-sup for $g \to e_G$ and $h \to e_H$ on both members, we get

$$\limsup_{(g,h)\to(e_G,e_H)} \|(\alpha \textcircled{w}_\gamma \beta)_{(g,h)}(x) - x\|_\gamma \leq 2\varepsilon\,,$$

since, for $j = 1, \dots, n$,

$$\limsup_{(g,h)\to(e_G \times e_H)} (\chi_{\partial a_j}(g) \chi_{\partial b_j}(h)) = \lim_{g \to e_G} \chi_{\partial a_j}(g) \lim_{h \to e_H} \chi_{\partial b_j}(h) = 1\,,$$

and the proof follows as $\varepsilon > 0$ is arbitrary. $\square$

Now, let $\mathcal{S}(\mathfrak{A}_o \mathbin{\textcircled{a}} \mathfrak{B}_o)$ be the family of all positive, unital, linear forms on $\mathfrak{A}_o \mathbin{\textcircled{a}} \mathfrak{B}_o$ that is, under the notation in Section II.10,

$$\mathcal{S}(\mathfrak{A}_o \mathbin{\textcircled{a}} \mathfrak{B}_o) = \left\{ f \in (\mathfrak{A}_o \mathbin{\textcircled{a}} \mathfrak{B}_o)' \colon f|_{\sum(\mathfrak{A}_o \mathbin{\textcircled{a}} \mathfrak{B}_o)^2} \geq 0,\ f(\mathbb{1}_{\mathfrak{A}} \mathbin{\textcircled{a}} \mathbb{1}_{\mathfrak{B}}) = 1 \right\} .$$

Notice that (the restriction of) the transpose map δ^{t} of each *-automorphism δ of $\mathfrak{A}_o \mathbin{\textcircled{a}} \mathfrak{B}_o$ induces a map

$$\mathcal{S}(\mathfrak{A}_o \mathbin{\textcircled{a}} \mathfrak{B}_o) \supset S \mapsto \delta^{\mathrm{t}}(S) \subset \mathcal{S}(\mathfrak{A}_o \mathbin{\textcircled{a}} \mathfrak{B}_o)$$

sending separating families to separating families.
By [17], each $f \in \mathcal{S}(\mathfrak{A}_o \mathbin{\textcircled{a}} \mathfrak{B}_o)$ generates a (unique, up to unitary equivalence) cyclic representation $(\mathcal{D}_f, \pi_{f,o}, \xi_f)$ of the involutive algebra $\mathfrak{A}_o \mathbin{\textcircled{a}} \mathfrak{B}_o$ which, by Lemma II.3.1, must consist of bounded operators. By setting $\mathcal{H}_f := \overline{\mathcal{D}_f}$ and $\pi_f := \overline{\pi_{f,o}}$, we get the usual GNS triplet $(\mathcal{H}_f, \pi_f, \xi_f)$ associated to f. Notice that $\ker(\pi_f) \subset \mathfrak{n}_f \subset \mathfrak{A}_o \mathbin{\textcircled{a}} \mathfrak{B}_o$. It follows that, for every non-empty subfamily $S \subset \mathcal{S}(\mathfrak{A}_o \mathbin{\textcircled{a}} \mathfrak{B}_o)$,

$$p_S(\cdot) := \sup_{f \in S} \|\pi_f(\cdot)\|_{\mathcal{B}(\mathcal{H}_f)}$$

defines a C^*-seminorm on $\mathfrak{A}_o \mathbin{\textcircled{a}} \mathfrak{B}_o$ where, for every $y \in \mathfrak{A}_o \mathbin{\textcircled{a}} \mathfrak{B}_o$, the supremum is bounded by $\sqrt{C_y}$ found in the proof of Proposition II.10.1. If $S \subset \mathcal{S}(\mathfrak{A}_o \mathbin{\textcircled{a}} \mathfrak{B}_o)$ is a separating family, then p_S is a C^*-norm.
Conversely, every C^*-norm on $\mathfrak{A}_o \mathbin{\textcircled{a}} \mathfrak{B}_o$ is determined by some separating family of states $S \subset \mathcal{S}(\mathfrak{A}_o \mathbin{\textcircled{a}} \mathfrak{B}_o)$.
The situation, which takes into account the case of compatible norms as well, is summarized in the following

Theorem II.10.5
Every separating family of states $S \subset \mathcal{S}(\mathfrak{A}_o \mathbin{\textcircled{a}} \mathfrak{B}_o)$ determines a C^*-norm γ_S on $\mathfrak{A}_o \mathbin{\textcircled{a}} \mathfrak{B}_o$. If in addition $(\alpha_g \times \beta_h)^{\mathrm{t}}(S) = S$ for each $g \in G$ and $h \in H$, then γ_S is compatible. Conversely, there exist two injections

$$\mathcal{G} := \{\gamma \colon \gamma\ C^*\text{-norm on } \mathfrak{A}_o \mathbin{\textcircled{a}} \mathfrak{B}_o\} \hookrightarrow \{S \subset \mathcal{S}(\mathfrak{A}_o \mathbin{\textcircled{a}} \mathfrak{B}_o) \colon S \text{ convex, separating}\} ,$$
$$\mathcal{G}_{\mathrm{c}} := \{\gamma \colon \gamma \text{ compatible } C^*\text{-norm on } \mathfrak{A}_o \mathbin{\textcircled{a}} \mathfrak{B}_o\} \hookrightarrow \{S \subset \mathcal{S}_{G\times H}(\mathfrak{A}_o \mathbin{\textcircled{a}} \mathfrak{B}_o) \colon S \text{ convex, separating}\} .$$

Proof.
We have already seen that any separating class $S \subset \mathcal{S}(\mathfrak{A}_o \mathbin{\textcircled{a}} \mathfrak{B}_o)$ provides a C^*-norm γ_S of $\mathfrak{A}_o \mathbin{\textcircled{a}} \mathfrak{B}_o$. Suppose now that S is globally invariant under $\alpha_g \times \beta_h$ for each $g \in G$ and $h \in H$. Thanks to the compatibility of γ_S, we get

$$\begin{aligned} \|(\alpha_g \times \beta_h)(x)\|_{\gamma_S} &= \sup_{f \in S} \left\|\pi_f \circ (\alpha_g \times \beta_h)(x)\right\| = \sup_{f \in S} \left\|\pi_{(\alpha_g\times\beta_h)^{\mathrm{t}}(f)}(x)\right\| \\ &= \sup_{f \in (\alpha_g\times\beta_h)^{\mathrm{t}}(S)} \left\|\pi_f(x)\right\| = \sup_{f \in S} \left\|\pi_f(x)\right\| = \|x\|_{\gamma_S}\ (x \in \mathfrak{A}_o \mathbin{\textcircled{a}} \mathfrak{B}_o,\ g \in G,\ h \in H). \end{aligned}$$

Concerning the converse part, for each $\gamma \in \mathcal{G}$, we put $\mathfrak{C}_\gamma := \mathfrak{A} \mathbin{\textcircled{a}}_\gamma \mathfrak{B}$. The injection is given by

$$\mathcal{G} \ni \gamma \mapsto S_\gamma := \mathcal{S}(\mathfrak{C}_\gamma)|_{\mathfrak{A}_o \mathbin{\textcircled{a}} \mathfrak{B}_o} \subset \mathcal{S}(\mathfrak{A}_o \mathbin{\textcircled{a}} \mathfrak{B}_o)$$

Such a map is well defined since $\mathcal{S}(\mathfrak{C}_\gamma)$ is convex and separating for $\mathfrak{C}_\gamma$, so is S_γ. Moreover, it is injective: if $\gamma_1, \gamma_2 \in \mathcal{G}$ are such that $S_{\gamma_1}|_{\mathfrak{A}_o \mathbin{\textcircled{a}} \mathfrak{B}_o} = S_{\gamma_2}|_{\mathfrak{A}_o \mathbin{\textcircled{a}} \mathfrak{B}_o}$ then, by faithfulness of the universal GNS representation $\bigoplus_{\tilde{f} \in \mathcal{S}(\mathfrak{C}_\gamma)} \pi_{\tilde{f}}$ of $\mathfrak{C}_\gamma$, $\gamma \in \mathcal{G}$, for every $x \in \mathfrak{A}_o \mathbin{\textcircled{a}} \mathfrak{B}_o$ we get

$$\begin{aligned} \|x\|_{\gamma_1} &= \sup_{\tilde{f} \in \mathcal{S}(\mathfrak{C}_{\gamma_1})} \|\pi_{\tilde{f}}(x)\| = \sup_{f \in S_{\gamma_1}} \|\pi_f(x)\| \\ &= \sup_{f \in S_{\gamma_2}} \|\pi_f(x)\| = \sup_{\tilde{f} \in \mathcal{S}(\mathfrak{C}_{\gamma_2})} \|\pi_{\tilde{f}}(x)\| = \|x\|_{\gamma_2} , \end{aligned}$$

since for every $\widetilde{f} \in \mathcal{S}(\mathfrak{C}_\gamma)$, $\pi_{\widetilde{f}|_{\mathfrak{A}_o \textcircled{u} \mathfrak{B}_o}} = \pi_{\widetilde{f}}|_{\mathfrak{A}_o \textcircled{u} \mathfrak{B}_o}$, $\gamma \in \mathcal{G}$.
For the last part, we suppose that $\gamma \in \mathcal{G}_c$. This means that the product action $\alpha \times \beta$ of $G \times H$ on $\mathfrak{A}_o \textcircled{u} \mathfrak{B}_o$ extends to an action, denoted by $\alpha \textcircled{u}_\gamma \beta$, on the whole $\mathfrak{A} \textcircled{u}_\gamma \mathfrak{B}$ which leads to the C^*-system $(\mathfrak{A} \textcircled{u}_\gamma \mathfrak{B}, G \times H, \alpha \textcircled{u}_\gamma \beta)$ as stated in Proposition II.10.4. We note that the $G \times H$-invariant states $\mathcal{S}_{G\times H}(\mathfrak{A} \textcircled{u}_\gamma \mathfrak{B})$ separate the points. With $S_{(G\times H,\gamma)} := \mathcal{S}_{G\times H}(\mathfrak{A} \textcircled{u}_\gamma \mathfrak{B})|_{\mathfrak{A}_o \textcircled{u} \mathfrak{B}_o}$, such a class of states determines the same norm as γ on $\mathfrak{A}_o \textcircled{u} \mathfrak{B}_o$:

$$\|x\|_\gamma = \|x\|_{\gamma_{S_{(G\times H,\gamma)}}}, \quad x \in \mathfrak{A}_o \textcircled{u} \mathfrak{B}_o .$$

The proof ends as the map $\mathcal{G}_c \ni \gamma \mapsto S_{(G\times H,\gamma)}$ is the injection we are searching for. □

II.11. Maximal and minimal twisted C^*-tensor products

As for the non-twisted tensor product, we define the *maximal* (i.e. universal) and the *minimal* (i.e. spatial) C^*-norms. Indeed, for $x \in \mathfrak{A}_o \textcircled{u} \mathfrak{B}_o$,

$$\|x\|_{\max} := \sup\left\{\|\pi(x)\| : \pi \in \mathsf{Rep}(\mathfrak{A}_o \textcircled{u} \mathfrak{B}_o)\right\},$$
$$\|x\|_{\min} := \sup\left\{\|\pi_{\omega\times\varphi}(x)\| : \omega \in \mathcal{S}_G(\mathfrak{A}),\ \ \varphi \in \mathcal{S}_H(\mathfrak{B})\right\}.$$

Obviously, $\|\ \|_{\min} \leq \|\ \|_{\max}$ as already happens in the usual case.

Remark II.11.1
It is possible to express the completions w.r.t. such norms in a simple way by using the associated representations:

$$\mathfrak{A} \textcircled{u}_{\max} \mathfrak{B} = \overline{\left\{ \bigoplus_{\pi \in \mathsf{Rep}(\mathfrak{A}_o \textcircled{u} \mathfrak{B}_o)} \pi(x) : x \in \mathfrak{A}_o \textcircled{u} \mathfrak{B}_o \right\}},$$
$$\mathfrak{A} \textcircled{u}_{\min} \mathfrak{B} = \overline{\left\{ \bigoplus_{\substack{\omega \in \mathcal{S}_G(\mathfrak{A}) \\ \varphi \in \mathcal{S}_H(\mathfrak{B})}} \pi_{\omega\times\varphi}(x) : x \in \mathfrak{A}_o \textcircled{u} \mathfrak{B}_o \right\}}.$$

Since the greatest norm $\|\ \|_{\max}$ arises from the universal representation of $\mathfrak{A}_o \textcircled{u} \mathfrak{B}_o$, it is also called *universal*. Its universal property is stated in Theorem II.14.1.
In Section II.12, we will see that the min-norm is indeed minimal among all the compatible norms, and in Section II.15 why it is called *spatial*.

Proposition II.11.2
The C^*-norms max and min are compatible.

Proof.
Notice that the maximal and the minimal norm are associated to the following classes of states, the universal one $\mathcal{S}(\mathfrak{A}_o \textcircled{u} \mathfrak{B}_o)$ and the class $\mathcal{S}_G(\mathfrak{A}) \times \mathcal{S}_H(\mathfrak{B})$ made of product states of invariant ones, respectively. Since both are left globally stable by the transposed action $(\alpha \times \beta)^{\mathrm{t}}$, the proof directly follows from Theorem II.10.5. □

The extension of the product action of $G \times H$ on the previous completions shall be denoted by $\alpha \textcircled{u}_{\max} \beta$ and $\alpha \textcircled{u}_{\min} \beta$, respectively.

Lemma II.11.3
The max-norm is sub-cross.

Proof.
Since the product states $\omega \times \varphi$ (with $\omega \in \mathcal{S}_G(\mathfrak{A})$ and $\varphi \in \mathcal{S}_H(\mathfrak{B})$) extend to states on $\mathfrak{A} ⓦ_{\max} \mathfrak{B}$ which are invariant w.r.t. $\alpha ⓦ_{\max} \beta$, we have $\mathcal{S}_{G\times H}(\mathfrak{A} ⓦ_{\max} \mathfrak{B})|_{\mathfrak{A}_o ⓦ \mathbb{1}_{\mathfrak{B}}} = \mathcal{S}_G(\mathfrak{A})|_{\mathfrak{A}_o}$ and, analogously, for the restriction to $\mathbb{1}_{\mathfrak{A}} ⓦ \mathfrak{B}_o$. We have already noticed that the action $\alpha ⓦ_{\max} \beta$ is pointwise norm-continuous, and thus the corresponding invariant states separate the points of $\mathfrak{A} ⓦ_{\max} \mathfrak{B}$, as $\mathcal{S}_G(\mathfrak{A})$ and $\mathcal{S}_H(\mathfrak{G})$ do for $\mathfrak{A}$ and $\mathfrak{B}$, respectively. Consequently, for $a \in \mathfrak{A}_o$,

$$\|a ⓦ \mathbb{1}_{\mathfrak{B}}\|_{\max} = \sup_{\psi \in \mathcal{S}_{G\times H}(\mathfrak{A} ⓦ_{\max} \mathfrak{B})} \|\pi_\psi(a ⓦ \mathbb{1}_{\mathfrak{B}})\| = \sup_{\omega \in \mathcal{S}_G(\mathfrak{A})} \|\pi_\omega(a)\| = \|a\|_{\mathfrak{A}}$$

and, analogously, $\|\mathbb{1}_{\mathfrak{A}} ⓦ b\|_{\max} = \|b\|_{\mathfrak{B}}$ for $b \in \mathfrak{B}_o$. Therefore, for each $a \in \mathfrak{A}_o$ and $b \in \mathfrak{B}_o$,

$$\|a ⓦ b\|_{\max} = \|(a ⓦ \mathbb{1}_{\mathfrak{B}})(\mathbb{1}_{\mathfrak{A}} ⓦ b)\|_{\max} \leq \|a ⓦ \mathbb{1}_{\mathfrak{B}}\|_{\max} \|\mathbb{1}_{\mathfrak{A}} ⓦ b\|_{\max} = \|a\|_{\mathfrak{A}} \|b\|_{\mathfrak{B}} \square$$

Remark II.11.4
Notice that:

(i) exactly the same proof as above allows to conclude that each compatible C^*-norm is sub-cross;

(i') (i) can be strengthened by asserting that each C^*-norm is sub-cross;

(ii) each representation π of $\mathfrak{A}_o ⓦ \mathfrak{B}_o$ is separately continuous.

Proof.
For (i'), we fix any C^*-norm γ and apply the following Proposition II.11.5 (the proof of which relies on the sub-cross property of the max-norm established in the previous lemma), obtaining for $a \in \mathfrak{A}_o$ and $b \in \mathfrak{B}_o$

$$\begin{aligned} \|a ⓦ b\|_\gamma =& \|\iota_{\mathfrak{A}}(a)\iota_{\mathfrak{B}}(b)\|_\gamma \leq \|\iota_{\mathfrak{A}}(a)\iota_{\mathfrak{B}}(b)\|_{\max} \\ \leq& \|\iota_{\mathfrak{A}}(a)\|_{\max} \|\iota_{\mathfrak{B}}(b)\|_{\max} \leq \|a\|_{\mathfrak{A}} \|b\|_{\mathfrak{B}} \,. \end{aligned}$$

For (ii), since $\pi|_{\mathfrak{A}_o ⓦ \mathbb{1}_{\mathfrak{B}}}$ and $\pi|_{\mathbb{1}_{\mathfrak{A}} ⓦ \mathfrak{B}_o}$ are representations of $\mathfrak{A}_o$ and $\mathfrak{B}_o$, respectively, the assertion follows from Proposition II.5.3. $\square$

The forthcoming result concerns the extensions of the injections

$$\begin{aligned} \mathfrak{A}_o \ni a \mapsto \iota_{\mathfrak{A}}(a) := a ⓦ \mathbb{1}_{\mathfrak{B}} \in \mathfrak{A} ⓦ_\gamma \mathfrak{B} \,, \\ \mathfrak{B}_o \ni b \mapsto \iota_{\mathfrak{B}}(b) := \mathbb{1}_{\mathfrak{A}} ⓦ b \in \mathfrak{A} ⓦ_\gamma \mathfrak{B} \,, \end{aligned}$$

where γ is any C^*-norm. In particular, it tells us that, if the norm γ is cross, $\mathfrak{A} ⓦ_\gamma \mathfrak{B}$ contains isomorphic copies of $\mathfrak{A}$ and $\mathfrak{B}$ as desired.

Proposition II.11.5
For each C^*-norm γ on $\mathfrak{A}_o ⓦ \mathfrak{B}_o$, the maps

$$\begin{aligned} \iota^\gamma_{\mathfrak{A}} \colon (\mathfrak{A}, \|\ \|_{\mathfrak{A}}) &\to \mathfrak{A} ⓦ_\gamma \mathfrak{B} \\ a &\mapsto \iota^\gamma_{\mathfrak{A}}(a) := \lim_n (a_n ⓦ \mathbb{1}_{\mathfrak{B}}), \quad (a_n)_n \subset \mathfrak{A}_o \text{ s.t. } \lim_n a_n = a \,, \\ \iota^\gamma_{\mathfrak{B}} \colon (\mathfrak{B}, \|\ \|_{\mathfrak{B}}) &\to \mathfrak{A} ⓦ_\gamma \mathfrak{B} \,; \\ b &\mapsto \iota^\gamma_{\mathfrak{B}}(b) := \lim_n (\mathbb{1}_{\mathfrak{A}} ⓦ b_n), \quad (b_n)_n \subset \mathfrak{B}_o \text{ s.t. } \lim_n b_n = b \end{aligned}$$

are well defined *-homomorphisms.
If γ is cross, then $\iota^\gamma_{\mathfrak{A}}$ and $\iota^\gamma_{\mathfrak{B}}$ are isometric, and thus $\mathfrak{A}$ and $\mathfrak{B}$ are identified with two unital C^*-subalgebras $\iota^\gamma_{\mathfrak{A}}(\mathfrak{A}), \iota^\gamma_{\mathfrak{B}}(\mathfrak{B})$ of the completion $\mathfrak{A} ⓦ_\gamma \mathfrak{B}$.

Proof.
We deal with $\iota_{\mathfrak{A}}$ only, $\iota_{\mathfrak{B}}$ being similar.
We first note that the max-norm is sub-cross (cf. Lemma II.11.3), and that any norm γ is provided with a set of states S_γ which separates the points of $\mathfrak{A}_o \textcircled{w} \mathfrak{B}_o$ (cf. Theorem II.10.5).
Now, for each $a \in \mathfrak{A}$ we choose a sequence $(a_n)_n \subset \mathfrak{A}_o$ converging to a: $\|a - a_n\|_{\mathfrak{A}} \to 0$.
We claim that $(a_n \textcircled{w} \mathbb{1}_{\mathfrak{B}})_n \subset \mathfrak{A}_o \textcircled{w}_\gamma \mathfrak{B}_o$ is a Cauchy sequence. Indeed, for each representation π and $n, m \in \mathbb{N}$, we get

$$\begin{aligned}\|\pi(a_m \textcircled{w} \mathbb{1}_{\mathfrak{B}}) - \pi(a_n \textcircled{w} \mathbb{1}_{\mathfrak{B}})\| =& \big\|\pi\big((a_m - a_n) \textcircled{w} \mathbb{1}_{\mathfrak{B}}\big)\big\| \\ &\leq \|(a_m - a_n) \textcircled{w} \mathbb{1}_{\mathfrak{B}}\|_{\max} = \|a_m - a_n\|_{\mathfrak{A}} \,.\end{aligned}$$

In particular,

$$\begin{aligned}\|(a_m \textcircled{w} \mathbb{1}_{\mathfrak{B}}) - (a_n \textcircled{w} \mathbb{1}_{\mathfrak{B}})\|_\gamma &= \sup_{f \in S_\gamma} \big\|\pi_f\big((a_m \textcircled{w} \mathbb{1}_{\mathfrak{B}}) - (a_n \textcircled{w} \mathbb{1}_{\mathfrak{B}})\big)\big\| \\ &\leq \|a_m - a_n\|_{\mathfrak{A}} \,.\end{aligned}$$

Define $\iota^\gamma_{\mathfrak{A}}(a) := \lim_n (a_n \textcircled{w} \mathbb{1}_{\mathfrak{B}})$. It is a matter of routine to verify firstly that the limit does not depend on the chosen sequence, and secondly that $\iota^\gamma_{\mathfrak{A}}$ results to be a *-homomorphism whose range is necessarily a C^*-subalgebra.
Suppose now that γ is a cross norm. Then

$$\|\iota^\gamma_{\mathfrak{A}}(a)\|_\gamma = \lim_n \|a_n \textcircled{w} \mathbb{1}_{\mathfrak{B}}\|_\gamma = \lim_n (\|a_n\|_{\mathfrak{A}} \|\mathbb{1}_{\mathfrak{B}}\|_{\mathfrak{B}}) = \lim_n \|a_n\|_{\mathfrak{A}} = \|a\|_{\mathfrak{A}} \,.$$

Therefore, $\iota^\gamma_{\mathfrak{A}}$ is isometric, hence it is a *-monomorphism. □

We now exhibit a simple case for which the min and max C^*-norms coincide.

Proposition II.11.6
Suppose that $(\mathfrak{A}, G, \alpha)$ and $(\mathfrak{B}, H, \beta)$ are ergodic. Then, $\mathfrak{A} \textcircled{w}_{\max} \mathfrak{B}$ admits a faithful state $\omega_{\alpha \textcircled{w}_{\max} \beta} = \omega_\alpha \times \omega_\beta$. Consequently, $\mathfrak{A} \textcircled{w}_{\max} \mathfrak{B} = \mathfrak{A} \textcircled{w}_{\min} \mathfrak{B}$.
If in addition $\omega_\alpha \in \mathcal{S}_G(\mathfrak{A})$ and $\omega_\beta \in \mathcal{S}_G(\mathfrak{B})$ are tracial states, $\omega_{\alpha \textcircled{w}_{\max} \beta}$ is a trace.[5]

Proof.
We have already seen that, under completion w.r.t. the maximal norm, we get the C^*-system $(\mathfrak{A} \textcircled{w}_{\max} \mathfrak{B}, G \times H, \alpha \textcircled{w}_{\max} \beta)$. Since α and β are supposed to be ergodic, $\alpha \textcircled{w}_{\max} \beta$ is also ergodic and thus, as mentioned in Section II.4, there is a faithful state on $\mathfrak{A} \textcircled{w}_{\max} \mathfrak{B}$, which is the product state $\omega_\alpha \times \omega_\beta$. It is easy to verify that such a state is a trace if ω_α and ω_β are tracial states. Now, if $x \in \mathfrak{A}_o \textcircled{w} \mathfrak{B}_o$, then $\|x\|_{\min} \leq \|x\|_{\max} = \big\|\pi_{\omega_\alpha \times \omega_\beta}(x)\big\| \leq \|x\|_{\min}$. □

At this stage, we immediately prove the following

Proposition II.11.7
The max-norm and the min-norm are cross C^*-norms according to Definition II.10.2.

Proof.
Take a generic $a \in \mathfrak{A}_o$, and a homogeneous $b \in \mathfrak{B}_o$. For every fixed couple of states $\omega \in \mathcal{S}_G(\mathfrak{A})$ and $\varphi \in \mathcal{S}_H(\mathfrak{B})$, we use the representation in (II.13) to describe the GNS representation associated to the product state $\omega \times \varphi$. We then get

$$\begin{aligned}\|\pi_\omega(a)\|\|\pi_\varphi(b)\| =& \|(\pi_\omega(a) U(g_{\partial b}) \otimes \pi_\varphi(b)\| = \|\pi_{\omega \times \varphi}(a \textcircled{w} b)\| \\ \leq& \|a \textcircled{w} b\|_{\min} \leq \|a \textcircled{w} b\|_{\max} \leq \|a\|_{\mathfrak{A}} \|b\|_{\mathfrak{B}} \,,\end{aligned}$$

[5] Following the notation in [85], such a tracial state is denoted by $\tau_{\alpha \textcircled{w}_{\max} \beta}$.

where the first equality holds as the norm on $\mathcal{B}(\mathcal{H}_\omega \otimes \mathcal{H}_\varphi)$ is cross, and the last inequality follows by Lemma II.11.3.

Now, by taking into account that the invariant states under G separate the points of $\mathfrak{A}$, and those invariant under H separate the points of $\mathfrak{B}$, after taking the suprema on the left on all such invariant states, we get

$$\|a\|_{\mathfrak{A}}\|b\|_{\mathfrak{B}} \leq \|a \mathbin{\text{ⓦ}} b\|_{\min} \leq \|a \mathbin{\text{ⓦ}} b\|_{\max} \leq \|a\|_{\mathfrak{A}}\|b\|_{\mathfrak{B}} \,,$$

and thus all above inequalities are indeed equalities.

Instead, if a is homogeneous, we can use the representation in (II.14) as GNS of $\omega \times \varphi$ obtaining the same result. □

Remark II.11.8

The same proof tells us that each C^*-norm lying between the min-norm and the max-norm is automatically cross.

II.12. Minimality of the min-norm

By definition, it is apparent that the maximal C^*-norm must be the greatest among all the C^*-norms, in particular among the compatible ones. We will deepen this fact in Section II.14. On the contrary, at this stage it is not yet clear whether the minimal C^*-norm is indeed minimal among all the compatible ones. The aim of the present section is to prove this fact. In order to introduce the necessary notation, we take the chance to give a corrected version and proof of Proposition 4.10 in [19] (pp. 16-17)[6]. We start from a preparatory lemma, which is nothing but an excerpt of the proof of Lemma IV.4.18 in [104] (p. 215): we shall provide it here below, for the convenience of the reader. As a premise, let $\mathcal{U}(\mathfrak{A})$ be the *unitary group* of $\mathfrak{A}$. It is a norm-closed subgroup of the *general linear group* $GL(\mathfrak{A})$ of $\mathfrak{A}$ consisting of its invertible elements and $GL(\mathfrak{A}) = \mathcal{U}(\mathfrak{A})GL_0(\mathfrak{A})$, $GL_0(\mathfrak{A})$ being the *principal component* (the connected component containing $\mathbb{1}_{\mathfrak{A}}$). $\mathcal{U}(\mathfrak{A})$ acts on the weakly-* compact, convex family of states $\mathcal{S}(\mathfrak{A})$ by affine homeomorphisms via the *adjoint action*:

$$\begin{aligned} \mathrm{ad}\colon \mathcal{U}(\mathfrak{A}) &\to \mathsf{Homeo}(\mathcal{S}(\mathfrak{A})) \\ u &\mapsto [\omega \mapsto \omega_u := \omega(u^* \cdot u)] \end{aligned}$$

In particular, the family of pure states $\mathcal{P}(\mathfrak{A}) \subseteq \mathcal{S}(\mathfrak{A})$ is ad-invariant. In presence of two C^*-algebras, $\mathfrak{A}$ and $\mathfrak{B}$, we canonically get an action of the direct product group $\mathcal{U}(\mathfrak{A}) \times \mathcal{U}(\mathfrak{B})$ on $\mathcal{S}(\mathfrak{A}) \times \mathcal{S}(\mathfrak{B})$:

$$\begin{aligned} \mathrm{ad} \times \mathrm{ad}\colon \mathcal{U}(\mathfrak{A}) \times \mathcal{U}(\mathfrak{B}) &\to \mathsf{Homeo}(\mathcal{S}(\mathfrak{A}) \times \mathcal{S}(\mathfrak{B})) \\ (u, v) &\mapsto [(\omega, \varphi) \mapsto (\omega_u, \varphi_v)] \end{aligned}$$

Again, $\mathcal{P}(\mathfrak{A}) \times \mathcal{P}(\mathfrak{B}) \subseteq \mathcal{S}(\mathfrak{A}) \times \mathcal{S}(\mathfrak{B})$ is $(\mathrm{ad} \times \mathrm{ad})$-invariant. Let

$$\mathcal{P}(\mathfrak{A}) \otimes \mathcal{P}(\mathfrak{B}) := \{\psi_{\omega,\varphi}\colon (\omega, \varphi) \in \mathcal{P}(\mathfrak{A}) \times \mathcal{P}(\mathfrak{B})\}$$

where $\psi_{\omega,\varphi}\colon \mathfrak{A} \otimes \mathfrak{B} \to \mathbb{C}$ is the (positive) product functional of ω and φ on $\mathfrak{A} \otimes \mathfrak{B}$. The following lemma shows that $\mathcal{P}(\mathfrak{A}) \otimes \mathcal{P}(\mathfrak{B})$ is minimal (w.r.t. the inclusion) among all the separating, invariant, weakly-* closed families of states on the *-algebra $\mathfrak{A} \otimes \mathfrak{B}$.

[6] That proposition in [19] relies on Lemma 4.9 (p. 16) which turns out to be wrong, as we will show in the forthcoming Section II.13.

Lemma II.12.1 (Takesaki)
Let $\mathfrak{A}, \mathfrak{B}$ two unital C^*-algebras. For every non-empty, $(\mathrm{ad} \times \mathrm{ad})$-invariant, $(\tau_{w^*} \times \tau_{w^*})$-closed subset $\mathcal{S} \subsetneq \mathcal{P}(\mathfrak{A}) \otimes \mathcal{P}(\mathfrak{B})$, there exists a pair of strictly positive $a \in \mathfrak{A}, b \in \mathfrak{B}$ s.t.

$$a \odot b \in \mathcal{S}^\perp := \bigcap_{\psi_{\omega,\varphi} \in \mathcal{S}} \ker(\psi_{\omega,\varphi})\,.$$

In particular, $\mathcal{S}$ is not separating for $\mathfrak{A} \otimes_{\min} \mathfrak{B}$.

Proof.
Let $S := \{(\omega,\varphi) \colon \psi_{\omega,\varphi} \in \mathcal{S}\} \subseteq \mathcal{P}(\mathfrak{A}) \times \mathcal{P}(\mathfrak{B})$. Let $U \subseteq \mathcal{P}(\mathfrak{A}), V \subseteq \mathcal{P}(\mathfrak{B})$ be non-empty τ_{w^*}-open sets s.t. $(U \times V) \cap S = \varnothing$. In particular, either $U \subsetneq \mathcal{P}(\mathfrak{A})$ or $V \subsetneq \mathcal{P}(\mathfrak{B})$. By $(\mathrm{ad} \times \mathrm{ad})$-invariance of S, their orbits under the adjoint action of the unitary groups $\mathcal{U}(\mathfrak{A})$ and $\mathcal{U}(\mathfrak{B})$ respectively, that is

$$\begin{cases} O_U := \mathrm{ad}_{\mathcal{U}(\mathfrak{A})}(U) = \{\omega(u^* \cdot u) \colon \omega \in U, u \in \mathcal{U}(\mathfrak{A})\} \\ O_V = \mathrm{ad}_{\mathcal{U}(\mathfrak{B})}(V) = \{\varphi(v^* \cdot v) \colon \varphi \in V, v \in \mathcal{U}(\mathfrak{B})\}\,, \end{cases}$$

are non-empty, ad-invariant, τ_{w^*}-open sets s.t. $(O_U \times O_V) \cap S = \varnothing$. Therefore, $K := O_U^c \subseteq \mathcal{P}(\mathfrak{A})$ and $L := O_V^c \subseteq \mathcal{P}(\mathfrak{B})$ are ad-invariant τ_{w^*}-closed sets (with either $K \neq \varnothing$ or $L \neq \varnothing$) s.t. $S \subseteq (K \times \mathcal{P}(\mathfrak{B})) \cup (\mathcal{P}(\mathfrak{A}) \times L)$. In particular, their *annihilators* (i.e. their orthogonal complements w.r.t. the dual pairings ${}_{\mathfrak{A}^*}\langle\cdot,\cdot\rangle_{\mathfrak{A}}$ and ${}_{\mathfrak{B}^*}\langle\cdot,\cdot\rangle_{\mathfrak{B}}$, respectively)

$$\begin{cases} K^\perp := \bigcap_{\omega \in K} \ker\omega \\ L^\perp := \bigcap_{\varphi \in L} \ker\varphi \end{cases}$$

are closed two-sided ideals of $\mathfrak{A}$ and $\mathfrak{B}$ respectively (see Lemma 4.15 in [104], p. 213). Since either $K^\perp$ or $L^\perp$ is non-zero, we always find a pair of strictly positive elements $a \in \mathfrak{A}, b \in \mathfrak{B}$ s.t. $\psi_{\omega,\varphi}(a \odot b) = 0$ for every $(\omega,\varphi) \in S$. It follows that $\mathcal{S}$ is not separating for $\mathfrak{A} \otimes_{\min} \mathfrak{B}$. □

Consider the completion $\mathfrak{A} \circledcirc_\gamma \mathfrak{B}$ of $\mathfrak{A}_o \circledcirc \mathfrak{B}_o$ w.r.t. a compatible norm γ. Recall that we have a C^*-system $(\mathfrak{A} \circledcirc_\gamma \mathfrak{B}, G \times H, \alpha \circledcirc_\gamma \beta)$. The following proposition asserts that, whenever the fixed point subalgebra of one of the marginal algebras is abelian, every extremal invariant state in $\mathcal{E}_{G\times H}(\mathfrak{A} \circledcirc_\gamma \mathfrak{B})$ of $\mathfrak{A} \circledcirc_\gamma \mathfrak{B}$ must look like an element of $\mathcal{E}_G(\mathfrak{A}) \times \mathcal{E}_H(\mathfrak{B})$ on the algebraic part $\mathfrak{A}_o \circledcirc \mathfrak{B}_o$. This result corrects Proposition 4.10 in [19].

Proposition II.12.2
If either $\mathfrak{A}^G$ or $\mathfrak{B}^H$ is abelian and γ is a compatible C^*-norm on $\mathfrak{A}_o \circledcirc \mathfrak{B}_o$, then there exists an injection

$$\begin{aligned} r \colon \mathcal{E}_{G\times H}(\mathfrak{A} \circledcirc_\gamma \mathfrak{B}) &\hookrightarrow \mathcal{E}_G(\mathfrak{A}) \times \mathcal{E}_H(\mathfrak{B}) \\ \psi &\mapsto \big(E_G^{\mathrm{t}}(\psi_{\mathfrak{A}}), E_H^{\mathrm{t}}(\psi_{\mathfrak{B}})\big) \end{aligned}$$

where $\psi_{\mathfrak{A}} := \psi|_{\mathfrak{A}^G \circledcirc \mathbb{1}_{\mathfrak{B}}}(\,\cdot \circledcirc \mathbb{1}_{\mathfrak{B}})$ and $\psi_{\mathfrak{B}} := \psi|_{\mathbb{1}_{\mathfrak{A}} \circledcirc \mathfrak{B}^H}(\mathbb{1}_{\mathfrak{A}} \circledcirc \cdot\,)$. As a consequence, $\|\ \|_\gamma = \|\ \|_{\min}$.

Proof.
Define $\mathfrak{C}_\gamma := \mathfrak{A} \circledcirc_\gamma \mathfrak{B}$. We see at once that $\mathfrak{C}_\gamma^{G\times H} = \overline{\mathfrak{A}^G \circledcirc \mathfrak{B}^H}^{\gamma} = \mathfrak{A}^G \otimes_\gamma \mathfrak{B}^H = \mathfrak{A}^G \otimes_{\min} \mathfrak{B}^H$, the last equality due to the abelianness (hence, nuclearity) of either $\mathfrak{A}^G$ or $\mathfrak{B}^H$. Now, if $\psi \in \mathcal{E}_{G\times H}(\mathfrak{C}_\gamma)$, then $\psi|_{\mathfrak{A}^G \otimes_{\min} \mathfrak{B}^H}$ is pure. Therefore, by [104], Theorem IV.4.14 (p. 211), $\psi|_{\mathfrak{A}^G \otimes_{\min} \mathfrak{B}^H}$ coincides on the involutive algebra $\mathfrak{A}^G \otimes \mathfrak{B}^H$ with the product functional $\psi_{\mathfrak{A}} \odot \psi_{\mathfrak{B}}$, where

$$\psi_{\mathfrak{A}}(\cdot) := \psi|_{\mathfrak{A}^G \circledcirc \mathbb{1}_{\mathfrak{B}}}(\cdot \circledcirc \mathbb{1}_{\mathfrak{B}}) \in \mathcal{P}(\mathfrak{A}^G)\,,$$

$$\psi_{\mathfrak{B}}(\cdot) := \psi|_{\mathbb{1}_{\mathfrak{A}} \circledcirc \mathfrak{B}^H}(\mathbb{1}_{\mathfrak{A}} \circledcirc \cdot) \in \mathcal{P}(\mathfrak{B}^H)\,.$$

In particular, $E^{\mathrm{t}}_G(\psi_{\mathfrak{A}}) \in \mathcal{E}_G(\mathfrak{A})$, $E^{\mathrm{t}}_H(\psi_{\mathfrak{B}}) \in \mathcal{E}_H(\mathfrak{B})$ and

$$\psi|_{\mathfrak{A}_o \circledcirc \mathfrak{B}_o} = (\psi \circ E_{G\times H})\,|_{\mathfrak{A}_o \circledcirc \mathfrak{B}_o} = (\psi_{\mathfrak{A}} \circ E_G)|_{\mathfrak{A}_o} \times (\psi_{\mathfrak{B}} \circ E_H)|_{\mathfrak{B}_o} = E^{\mathrm{t}}_G(\psi_{\mathfrak{A}})|_{\mathfrak{A}_o} \times E^{\mathrm{t}}_H(\psi_{\mathfrak{B}})|_{\mathfrak{B}_o}$$

By density of $\mathfrak{A}_o \circledcirc \mathfrak{B}_o$ in $\mathfrak{C}_\gamma$, $r\colon \mathcal{E}_{G\times H}(\mathfrak{A} \circledcirc_\gamma \mathfrak{B}) \to \mathcal{E}_G(\mathfrak{A}) \times \mathcal{E}_H(\mathfrak{B})$ in the assertion is clearly injective. Moreover, since also the ergodic invariant states separate the points, we get for $x \in \mathfrak{A}_o \circledcirc \mathfrak{B}_o$,

$$\|x\|_\gamma = \sup_{\psi \in \mathcal{E}_{G\times H}(\mathfrak{C}_\gamma)} \|\pi_\psi(x)\| \leq \sup_{\substack{\omega \in \mathcal{E}_G(\mathfrak{A}) \\ \varphi \in \mathcal{E}_H(\mathfrak{B})}} \|\pi_\omega \circledcirc \pi_\varphi(x)\| = \|x\|_{\min}\,.$$

We are then left with proving that $\|\ \|_\gamma \geq \|\ \|_{\min}$, so that $\|\ \|_\gamma = \|\ \|_{\min}$. In view of Proposition II.3.2, it suffices to show that the product state $\omega \times \varphi$ is γ-bounded on $\mathfrak{A}_o \circledcirc \mathfrak{B}_o$ for any $\omega \in \mathcal{E}_G(\mathfrak{A}), \varphi \in \mathcal{E}_H(\mathfrak{B})$, so that by Proposition II.3.2

$$\|\ \|_{\min} = \sup_{\substack{\omega \in \mathcal{E}_G(\mathfrak{A}) \\ \varphi \in \mathcal{E}_H(\mathfrak{B})}} \|\pi_\omega \circledcirc \pi_\varphi(\cdot)\| \leq \|\ \|_\gamma\,.$$

Let us put

$$\begin{aligned} S_\gamma &:= \{(\omega,\varphi) \in \mathcal{E}_G(\mathfrak{A}) \times \mathcal{E}_H(\mathfrak{B})\colon \omega \times \varphi \ \gamma\text{-bounded on } \mathfrak{A}_o \circledcirc \mathfrak{B}_o\}\,, \\ \widetilde{S}_\gamma &:= \{(\widetilde{\omega},\widetilde{\varphi}) \in \mathcal{P}(\mathfrak{A}^G) \times \mathcal{P}(\mathfrak{B}^H)\colon \widetilde{\omega} \times \widetilde{\varphi} \ \gamma\text{-bounded on } \mathfrak{A}^G \otimes \mathfrak{B}^H\}\,. \end{aligned} \tag{II.16}$$

Both S_γ and $\widetilde{S}_\gamma$ are non-empty, invariant under the adjoint action of $\mathcal{U}(\mathfrak{A}^G) \times \mathcal{U}(\mathfrak{B}^H)$, $(\tau_{w^*} \times \tau_{w^*})$-closed subsets of $\mathcal{E}_G(\mathfrak{A}) \times \mathcal{E}_H(\mathfrak{B})$ and $\mathcal{P}(\mathfrak{A}^G) \times \mathcal{P}(\mathfrak{B}^H)$, respectively. It is also easy to verify that they are affinely homeomorphic via

$$\begin{aligned} S_\gamma \ni (\omega,\varphi) &\mapsto (\omega|_{\mathfrak{A}^G}, \varphi|_{\mathfrak{B}^H}) \in \widetilde{S}_\gamma\,, \\ S_\gamma \ni \big(E^{\mathrm{t}}_G(\widetilde{\omega}), E^{\mathrm{t}}_H(\widetilde{\varphi})\big) &\mapsfrom (\widetilde{\omega},\widetilde{\varphi}) \in \widetilde{S}_\gamma\,. \end{aligned}$$

We want to show that $S_\gamma = \mathcal{E}_G(\mathfrak{A}) \times \mathcal{E}_H(\mathfrak{B})$, which is then equivalent to show that $\widetilde{S}_\gamma = \mathcal{P}(\mathfrak{A}^G) \times \mathcal{P}(\mathfrak{B}^H)$. Suppose by contradiction that $S_\gamma \subsetneq \mathcal{E}_G(\mathfrak{A}) \times \mathcal{E}_H(\mathfrak{B})$, then $\widetilde{S}_\gamma \subsetneq \mathcal{P}(\mathfrak{A}^G) \times \mathcal{P}(\mathfrak{B}^H)$. By Lemma II.12.1, there exist strictly positive $a \in \mathfrak{A}^G, b \in \mathfrak{B}^H$ such that $a \odot b \in \widetilde{S}^\perp_\gamma$. This means that, also $a \odot b \in S^\perp_\gamma \subseteq \mathcal{E}_{G\times H}(\mathfrak{C}_\gamma)^\perp$, which is a contradiction, since $\mathcal{E}_{G\times H}(\mathfrak{C}_\gamma)$ must be separating. In conclusion, $S_\gamma = \mathcal{E}_G(\mathfrak{A}) \times \mathcal{E}_H(\mathfrak{B})$ and $\|\ \|_\gamma = \|\ \|_{\min}$. □

Theorem II.12.3
The spatial C^*-norm is minimal among all the compatible C^*-norms on $\mathfrak{A}_o \circledcirc \mathfrak{B}_o$.

Proof.
For a compatible norm γ on $\mathfrak{A}_o \circledcirc \mathfrak{B}_o$ and S_γ in (II.16), if $S_\gamma = \mathcal{E}_G(\mathfrak{A}) \times \mathcal{E}_H(\mathfrak{B})$, then $\|\ \|_{\min} \leq \|\ \|_\gamma$. It is then sufficient to show this equality. Suppose by contradiction that $S_\gamma \subsetneq \mathcal{E}_G(\mathfrak{A}) \times \mathcal{E}_H(\mathfrak{B})$, or equivalently, $\widetilde{S}_\gamma \subsetneq \mathcal{P}(\mathfrak{A}^G) \times \mathcal{P}(\mathfrak{B}^H)$. By reasoning as in the proof of Theorem IV.4.19 in [104] (p. 216), we deduce that $\widetilde{S}_\gamma$ must coincide with $\mathcal{P}(\mathfrak{A}^G) \times \mathcal{P}(\mathfrak{B}^H)$ and, consequently, $S_\gamma = \mathcal{E}_G(\mathfrak{A}) \times \mathcal{E}_H(\mathfrak{B})$. □

II.13. On (non) compatible C^*-norms

As we will see in the next chapter, the study of symmetric states acting on the twisted C^*-tensor product of infinitely many copies of a single algebra will need to consider the action of a fixed group G acting diagonally on the chain (see, for instance, [31]). We take here the occasion to introduce the starting point of this construction, since it is useful to compare our definition of *compatibility* of a C^*-norm with the one given in [19].
We start from two C^*-systems $(\mathfrak{A}, G, \alpha)$ and $(\mathfrak{B}, G, \beta)$ based on the same acting group G, and build a new one given by $\big(\mathfrak{A} \,\text{ⓦ}_{\min}\, \mathfrak{B}, G, \delta^{(\alpha,\beta)}\big)$, where $\delta^{(\alpha,\beta)}$ is the *diagonal action* of G, given on elementary tensors by

$$\delta_g^{(\alpha,\beta)}(a \odot b) := \alpha_g(a) \,\text{ⓦ}\, \beta_g(b)\,, \quad g \in G,\ a \in \mathfrak{A}_o, b \in \mathfrak{B}_o\,.$$

This action is perfectly meaningful, as we have seen that the product action of $G \times G$ on $\mathfrak{A}_o \,\text{ⓦ}\, \mathfrak{B}_o$ (and *a fortiori* its restriction to the diagonal of $G \times G$) always extends to an action on $\mathfrak{A} \,\text{ⓦ}_{\min}\, \mathfrak{B}$ (cf. Proposition II.10.4 and Proposition II.11.2) making $\big(\mathfrak{A} \,\text{ⓦ}_{\min}\, \mathfrak{B}, G, \delta^{(\alpha,\beta)}\big)$ a full-fledged C^*-system. Definition 4.6 at p. 15 of [19] (where the authors address the special case $G = \mathbb{Z}_2$) calls a C^*-norm γ *compatible* if $\delta^{(\alpha,\beta)}$ is γ-isometric. At the end of the present section, we will give an example for which this definition is strictly weaker than ours, namely a C^*-norm which is compatible for the diagonal action of $\mathbb{Z}_2$ but not for the full action of the Klein 4-group $K_4 = \mathbb{Z}_2 \times \mathbb{Z}_2$. The existence of such a norm raises the question whether the min-norm introduced above is actually the smallest among all the C^*-norms which are compatible with the diagonal action only, as affirmed in [19]. It is still unclear to us if this question can be answered positively or not, but at least we point out the reason why the proof in [19] (Theorem 4.12, p. 17) is not acceptable. Indeed, it relies on Lemma 4.9 (p. 16)

"If $(\mathfrak{A}, \mathbb{Z}_2, \alpha)$, $(\mathfrak{B}, \mathbb{Z}_2, \beta)$ are two C^*-systems, where either $\mathfrak{A}$ or $\mathfrak{B}$ is abelian, and $\mathbb{Z}_2 \overset{\delta^{(\alpha,\beta)}}{\curvearrowright} \mathfrak{A} \,\text{Ⓕ}\, \mathfrak{B}$ is γ-isometric for some C^*-norm γ, then $\mathcal{E}_{\mathbb{Z}_2}(\mathfrak{A} \,\text{Ⓕ}_\gamma\, \mathfrak{B}) \cong \mathcal{E}_{\mathbb{Z}_2}(\mathfrak{A}) \times \mathcal{E}_{\mathbb{Z}_2}(\mathfrak{B})$„

which is definitely wrong (its proof also contains a coarse mistake, when asserting that $\mathfrak{A} \,\text{Ⓕ}\, \mathbb{1}_{\mathfrak{B}}$ lies in the center of $\mathfrak{A} \,\text{Ⓕ}\, \mathfrak{B}$). We expose here an easy counterexample.[7]
Consider the Canonical Anticommutation Relations algebra (*CAR algebra*, for short) on two generators

$$\mathrm{CAR}(\{1,2\}) := C^*\left(a_j, a_j^\dagger \,\middle|\, a_j^* = a_j^\dagger, \{a_j, a_k\} = \{a_j^\dagger, a_k^\dagger\} = 0, \{a_j^\dagger, a_k\} = \delta_{jk}\mathbb{1},\ j,k = 1,2\right)$$

The parity automorphism $\vartheta \in \mathsf{Aut}\,(\mathrm{CAR}(\{1,2\}))$ defined on the generators by $\vartheta(a_j) := -a_j$ and $\vartheta(a_j^\dagger) := -a_j^\dagger$ $(j = 1,2)$ is involutive, hence yielding a $\mathbb{Z}_2$-grading on $\mathrm{CAR}(\{1,2\})$. Observe that if $U := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (or also $U := \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$) (unitary, selfadjoint element of $M_2(\mathbb{C})$), then

$$\alpha\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) := \mathrm{ad}_U\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & -b \\ -c & d \end{bmatrix}$$

is an involutive *-automorphism of $M_2(\mathbb{C})$. Therefore, $(M_2(\mathbb{C}) \,\text{Ⓕ}\, M_2(\mathbb{C}), \alpha \,\text{Ⓕ}\, \alpha)$ is a $\mathbb{Z}_2$-graded C^*-algebra and

$$\begin{aligned} \Phi\colon \mathrm{CAR}(\{1,2\}) &\to M_2(\mathbb{C}) \,\text{Ⓕ}\, M_2(\mathbb{C}) \\ a_1 &\mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \,\text{Ⓕ}\, I_2 \\ a_2 &\mapsto I_2 \,\text{Ⓕ}\, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

[7] This example was introduced in [1] for the investigation of the entanglement phenomenon in Fermi models. It was studied in a more detailed form in Section 11 of [31] to show that some crucial computations in [20] (even published much later than [31]) unfortunately contain fatal mistakes as well.

is a grading-equivariant *-isomorphism: $\Phi \circ \vartheta = (\alpha \textcircled{F} \alpha) \circ \Phi$. Let

$$\begin{cases} s := a_1 + a_1^\dagger, \quad S := a_2 + a_2^\dagger \in \mathrm{CAR}(\{1,2\})_- \\ p := \dfrac{\mathbb{1}+s}{2}, \quad q := \dfrac{\mathbb{1}-s}{2} \Rightarrow pq = 0 \text{ (mutually orthogonal projections in } \mathrm{CAR}(\{1,2\})) \\ P := \dfrac{\mathbb{1}+S}{2}, \quad Q := \dfrac{\mathbb{1}-S}{2} \Rightarrow PQ = 0 \text{ (mutually orthogonal projections in } \mathrm{CAR}(\{1,2\})) \\ \mathcal{A} := \langle p, q \rangle \overset{\Phi}{\cong} \left\langle \dfrac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \textcircled{F} I_2, \dfrac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \textcircled{F} I_2 \right\rangle \text{ (abelian } \vartheta\text{-invariant } C^*\text{-subalgebra)} \\ \mathcal{B} := \langle P, Q \rangle \overset{\Phi}{\cong} \left\langle I_2 \textcircled{F} \dfrac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, I_2 \textcircled{F} \dfrac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \right\rangle \text{ (abelian } \vartheta\text{-invariant } C^*\text{-subalgebra)} \end{cases}$$

(For a detailed account of this framework, see also Proposition 4.4 in [1], p. 172). Therefore,

$$\mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_- = \langle p+q \rangle \oplus \langle p-q \rangle = \langle \mathbb{1} \rangle \oplus \mathrm{span}_{\mathbb{C}}\{s\}$$
$$\mathcal{B} = \mathcal{B}_+ \oplus \mathcal{B}_- = \langle P+Q \rangle \oplus \langle P-Q \rangle = \langle \mathbb{1} \rangle \oplus \mathrm{span}_{\mathbb{C}}\{S\}$$

In particular, $\vartheta|_{\mathcal{A}}$ and $\vartheta|_{\mathcal{B}}$ are ergodic so $\mathcal{S}_\vartheta(\mathcal{A}) = \mathcal{E}_\vartheta(\mathcal{A}) = \{\mathrm{tr}_{\mathcal{A}}\}$ and $\mathcal{S}_\vartheta(\mathcal{B}) = \mathcal{E}_\vartheta(\mathcal{B}) = \{\mathrm{tr}_{\mathcal{B}}\}$ where

$$\mathrm{tr}_{\mathcal{A}} \colon \mathbb{1} \mapsto 1, \quad s \mapsto 0\,, \qquad \mathrm{tr}_{\mathcal{B}} \colon \mathbb{1} \mapsto 1, \quad S \mapsto 0$$

are the tracial states (normalized traces) of $\mathcal{A}$ and $\mathcal{B}$, respectively. Observe that $\mathrm{tr}_{\mathcal{A}}(p) = \mathrm{tr}_{\mathcal{A}}(q) = \mathrm{tr}_{\mathcal{B}}(P) = \mathrm{tr}_{\mathcal{B}}(Q) = 1/2$.

On the other hand, $(\langle \mathcal{A}, \mathcal{B} \rangle, \vartheta|_{\langle \mathcal{A},\mathcal{B} \rangle})$ is a $\mathbb{Z}_2$-graded, non-abelian, four-dimensional C^*-subalgebra of $\mathrm{CAR}(\{1,2\})$, isomorphic to $(\mathcal{A}, \vartheta|_{\mathcal{A}}) \textcircled{F} (\mathcal{B}, \vartheta|_{\mathcal{B}})$. Its spectral decomposition is

$$\begin{aligned} \langle \mathcal{A}, \mathcal{B} \rangle = \langle \mathcal{A}, \mathcal{B} \rangle_+ \oplus \langle \mathcal{A}, \mathcal{B} \rangle_- &= \langle \mathbb{1}, sS \rangle \oplus \mathrm{span}_{\mathbb{C}}\{s, S\} \\ &\overset{\Phi}{\cong} \langle I_2 \textcircled{F} I_2, \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] \textcircled{F} \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] \rangle \oplus \mathrm{span}_{\mathbb{C}} \left\{ \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] \textcircled{F} I_2, I_2 \textcircled{F} \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] \right\} . \end{aligned}$$

In particular, $\vartheta|_{\langle \mathcal{A},\mathcal{B} \rangle}$ is not ergodic and the mapping

$$\begin{aligned} \langle \mathcal{A}, \mathcal{B} \rangle_+ &\to \mathbb{C}^2 \\ e_1 := \frac{\mathbb{1} - isS}{2} = \frac{\mathbb{1} - 2i[p,P]}{2} &\mapsto (1,0) \\ e_2 := \frac{\mathbb{1} + isS}{2} = \frac{\mathbb{1} + 2i[p,P]}{2} &\mapsto (0,1) \end{aligned}$$

is a *-isomorphism of C^*-algebras. Therefore, $\mathcal{E}_\vartheta(\langle \mathcal{A}, \mathcal{B} \rangle) = \{e_1^* \circ E_+, e_2^* \circ E_+\} = \Omega_{\mathcal{C}(\{e_1,e_2\})} \circ E_+$ and

$$\mathcal{S}_\vartheta(\langle \mathcal{A}, \mathcal{B} \rangle) = \mathrm{co}(\mathcal{E}_\vartheta(\langle \mathcal{A}, \mathcal{B} \rangle)) = \{(\lambda e_1^* + (1-\lambda) e_2^*) \circ E_+ \colon \lambda \in [0,1]\}$$

We can also view e_1^*, e_2^* as pure states of $\langle \mathcal{A}, \mathcal{B} \rangle_+$ (equivalently, characters or irreducible representations of $\langle \mathcal{A}, \mathcal{B} \rangle_+$). Observe that $\mathrm{tr}_{\mathcal{A}} \times \mathrm{tr}_{\mathcal{B}} \in \mathcal{S}_\vartheta(\langle \mathcal{A}, \mathcal{B} \rangle)$ is not ergodic for this action. Precisely, it is the midpoint of the two extremal invariant states:

$$\mathrm{tr}_{\mathcal{A}} \times \mathrm{tr}_{\mathcal{B}} = \frac{e_1^* \circ E_+ + e_2^* \circ E_+}{2}$$

In conclusion, $\mathcal{E}_\vartheta((\mathcal{A}, \vartheta|_{\mathcal{A}}) \textcircled{F} (\mathcal{B}, \vartheta|_{\mathcal{B}})) \cong \mathcal{E}_\vartheta(\langle \mathcal{A}, \mathcal{B} \rangle) \ncong \mathcal{E}_\vartheta(\mathcal{A}) \times \mathcal{E}_\vartheta(\mathcal{B})$: Lemma 4.9 in [19] is false.

At this point, we take advantage of an example given by Wassermann in an addendum at the end of [79], p. 69, in order to provide

- an example of non-compatible C^*-norm, according to our definition
- an example of C^*-norm which is compatible according to the definition in [19], but not to ours

We start with Wassermann's example.

Theorem II.13.1 (Wassermann, [79])
Let $\mathfrak{A}$ be a non-nuclear C^*-algebra and $\vartheta \in \mathsf{Aut}(\mathfrak{A} \oplus \mathfrak{A})$ the *flip* automorphism

$$\vartheta(a, b) := (b, a)\,, \quad a, b \in \mathfrak{A}\,.$$

Then, there exists a (necessarily non-nuclear) C^*-algebra $\mathfrak{B}$ and a C^*-norm $\|\ \|_\rho$ on the involutive algebra $(\mathfrak{A} \oplus \mathfrak{A}) \otimes \mathfrak{B}$ such that $\vartheta \otimes \mathrm{id}_{\mathfrak{B}} \in \mathrm{Aut}((\mathfrak{A} \oplus \mathfrak{A}) \otimes \mathfrak{B})$ is not $\|\ \|_\rho$-continuous.

Proof.
If the unique possible C^*-norm on $\mathfrak{A} \otimes \mathfrak{B}$ was the spatial one $\|\ \|_{\min}$ for every C^*-algebra $\mathfrak{B}$, then the C^*-algebra $\mathfrak{A}$ would be nuclear (see Theorem 3.8.7 in [88], p. 104). Therefore, there must exist a C^*-algebra $\mathfrak{B}$, necessarily non-nuclear, for which $\mathfrak{A} \otimes \mathfrak{B}$ admits a C^*-norm γ which is not equivalent to $\|\ \|_{\min}$ (i.e. $\|\ \|_{\min} \leq \|\ \|_\gamma$, but $\|\ \|_{\min} \neq \|\ \|_\gamma$). Consider the completion $\mathfrak{A} \otimes_\gamma \mathfrak{B}$ of $\mathfrak{A} \otimes \mathfrak{B}$ w.r.t. γ and the C^*-algebra $\mathfrak{C} := (\mathfrak{A} \otimes_\gamma \mathfrak{B}) \oplus (\mathfrak{A} \otimes_{\min} \mathfrak{B})$ endowed with the *direct sum* C^*-norm $\|(x, y)\|_\oplus := \|x\|_\gamma \vee \|y\|_{\min}$. Then, the involutive algebra $(\mathfrak{A} \otimes \mathfrak{B}) \oplus (\mathfrak{A} \otimes \mathfrak{B})$ is evidently dense in $\mathfrak{C}$ w.r.t. $\|\ \|_\oplus$.
On the other hand, the right factoring-out map R given in (II.1) is also a *-isomorphism of involutive algebras. Then, the *pushforward* norm $\|\cdot\|_\rho := \|R^{-1}(\cdot)\|_\oplus$ is a well defined C^*-norm on $(\mathfrak{A} \oplus \mathfrak{A}) \otimes \mathfrak{B}$, the completion of which is denoted by $(\mathfrak{A} \oplus \mathfrak{A}) \otimes_\rho \mathfrak{B}$. By construction, R isometrically extends to a *-isomorphism $(\mathfrak{C}, \|\ \|_\oplus) \to ((\mathfrak{A} \oplus \mathfrak{A}) \otimes_\rho \mathfrak{B}, \|\ \|_\rho)$ of C^*-algebras. With $\vartheta \in \mathsf{Aut}(\mathfrak{A} \oplus \mathfrak{A})$ the flip automorphism, we have

$$\begin{aligned}\left\|(\vartheta \otimes \mathrm{id}_{\mathfrak{B}})\Big(\sum_{i=1}^m (0, a_i) \otimes b_i\Big)\right\|_\rho &= \left\|\sum_{i=1}^m (a_i, 0) \otimes b_i\right\|_\rho \\ &= \left\|R^{-1}\Big(\sum_{i=1}^m (a_i, 0) \otimes b_i\Big)\right\|_\oplus = \left\|\Big(\sum_{i=1}^m a_i \otimes b_i, 0\Big)\right\|_\oplus = \left\|\sum_{i=1}^m a_i \otimes b_i\right\|_\gamma .\end{aligned}$$

On the other hand,

$$\begin{aligned}\left\|\sum_{i=1}^m (0, a_i) \otimes b_i\right\|_\rho &= \left\|R^{-1}\Big(\sum_{i=1}^m (0, a_i) \otimes b_i\Big)\right\|_\oplus \\ &= \left\|\Big(0, \sum_{i=1}^m a_i \otimes b_i\Big)\right\|_\oplus = \left\|\sum_{i=1}^m a_i \otimes b_i\right\|_{\min} .\end{aligned}$$

Summarising, with

$$x := \sum_{i=1}^m a_i \otimes b_i \in \mathfrak{A} \otimes \mathfrak{B}\,, \quad y := \sum_{i=1}^m (0, a_i) \otimes b_i \in (\mathfrak{A} \oplus \mathfrak{A}) \otimes \mathfrak{B}\,,$$

we have proved that

$$\|(\vartheta \otimes \mathrm{id}_{\mathfrak{B}})(y)\|_\rho = \|x\|_\gamma\,, \quad \|y\|_\rho = \|x\|_{\min}\,. \tag{II.17}$$

By contradiction, suppose that $\vartheta \otimes \mathrm{id}_{\mathfrak{B}}$ is $\|\ \|_\rho$-continuous. For each $x \in \mathfrak{A} \otimes \mathfrak{B}$, by (II.17) we get

$$\|x\|_{\min} \leq \|x\|_\gamma = \|(\vartheta \otimes \mathrm{id}_{\mathfrak{B}})(y)\|_\rho \leq \|y\|_\rho = \|x\|_{\min}$$

that is $\|x\|_{\min} = \|x\|_\gamma$: a contradiction. □

We now explain why the previous result implies that non-compatible C^*-norms on $\mathfrak{A}_o \circledcirc \mathfrak{B}_o$ may well exist in our setting. Indeed, for $\mathfrak{A}$ and $\mathfrak{B}$ as above, it is enough to consider the twisted product $(\mathfrak{A}\oplus\mathfrak{A}) \circledcirc \mathfrak{B}$ where, on $\mathfrak{A}\oplus\mathfrak{A}$ the acting group is $G \equiv \mathbb{Z}_2$ via the flip $\alpha_1 := \vartheta$, whereas on $\mathfrak{B}$ any compact abelian group H is acting through the trivial action $\beta_h := \mathrm{id}_{\mathfrak{B}}$ $(h \in H)$. Lastly, one can choose any bicharacter on $\widehat{\mathbb{Z}_2} \times \widehat{H} \cong \mathbb{Z}_2 \times \widehat{H}$. Since in such a situation we have $(\mathfrak{A}\oplus\mathfrak{A}) \circledcirc \mathfrak{B} = (\mathfrak{A}\oplus\mathfrak{A}) \otimes \mathfrak{B}$, we conclude that the former admits a non-compatible norm. For an explicit example of the previous theorem, take the non-nuclear C^*-algebra $\mathfrak{A} := \mathcal{B}(\ell^2)$. By Junge-Pisier theorem (see Theorem 13.5.1 in [88], p. 388), $\mathfrak{B} := \mathcal{B}(\ell^2)$ is s.t. $\|\cdot\|_{\max,\otimes}$ is inequivalent to $\|\cdot\|_{\min,\otimes}$ on $\mathfrak{A}\otimes\mathfrak{B}$. Since $\|\cdot\|_{\oplus}$ in the previous theorem coincides with $\|\cdot\|_{\max,\otimes}$ on $(\mathcal{B}(\ell^2)\otimes_{\max}\mathcal{B}(\ell^2)) \oplus (\mathcal{B}(\ell^2)\otimes_{\min}\mathcal{B}(\ell^2))$, $\|\cdot\|_\rho := \|R^{-1}(\cdot)\|_{\max,\otimes}$ is an incompatible C^*-norm on $(\mathcal{B}(\ell^2)\oplus\mathcal{B}(\ell^2),\vartheta) \circledcirc (\mathcal{B}(\ell^2), H, \beta)$.

Suitably modifying Wassermann's example, it might also be shown that there are norms which are compatible with the diagonal action (provided $G = H$) but not with the full action of $G\times G$. This means that our definition of compatible norm is indeed different from the one originally given in [19] for $G = H = \mathbb{Z}_2$.

Indeed, with the same notation as in Theorem II.13.1, let

$$\mathfrak{C} := (\mathfrak{A}\otimes_\gamma\mathfrak{B}) \oplus (\mathfrak{A}\otimes_{\min}\mathfrak{B}) \oplus (\mathfrak{A}\otimes_{\min}\mathfrak{B}) \oplus (\mathfrak{A}\otimes_\gamma\mathfrak{B})\,.$$

Then, the following combination of the factoring-out mappings (cf. (II.1))

$$L(R\oplus R)\colon \bigoplus_{k=1}^{4}(\mathfrak{A}\otimes\mathfrak{B}) \to (\mathfrak{A}\oplus\mathfrak{A})\otimes(\mathfrak{B}\oplus\mathfrak{B})\,,$$

induces a C^*-norm ρ on $(\mathfrak{A}\oplus\mathfrak{A})\otimes(\mathfrak{B}\oplus\mathfrak{B})$ for which $\vartheta_{\mathfrak{A}\oplus\mathfrak{A}}\otimes\vartheta_{\mathfrak{B}\oplus\mathfrak{B}}$ is isometric, but $\vartheta_{\mathfrak{A}\oplus\mathfrak{A}}\otimes\mathrm{id}_{\mathfrak{B}\oplus\mathfrak{B}}$ and $\mathrm{id}_{\mathfrak{A}\oplus\mathfrak{A}}\otimes\vartheta_{\mathfrak{B}\oplus\mathfrak{B}}$ are not. In other words, ρ is compatible with the diagonal action $\mathbb{Z}_2 \overset{d}{\curvearrowright} \mathfrak{A}^{\oplus 2}\otimes\mathfrak{B}^{\oplus 2}$ but not with the full action $K_4 = \mathbb{Z}_2\times\mathbb{Z}_2 \overset{f}{\curvearrowright} \mathfrak{A}^{\oplus 2}\otimes\mathfrak{B}^{\oplus 2}$. The respective fixed point subalgebras are

$$(\mathfrak{A}^{\oplus 2}\otimes\mathfrak{B}^{\oplus 2})^{\mathbb{Z}_2} = \mathrm{span}_{\mathbb{C}}\{(a,a)\otimes(b,b)\colon a\in\mathfrak{A}, b\in\mathfrak{B}\} \oplus \mathrm{span}_{\mathbb{C}}\{(a,-a)\otimes(b,-b)\colon a\in\mathfrak{A}, b\in\mathfrak{B}\}\,,$$

$$(\mathfrak{A}^{\oplus 2}\otimes\mathfrak{B}^{\oplus 2})^{K_4} = \mathrm{span}_{\mathbb{C}}\{(a,a)\otimes(b,b)\colon a\in\mathfrak{A}, b\in\mathfrak{B}\}\,.$$

II.14. Characterizations of the max-norm

We begin this section with the following pivotal result: the universal property of the max-norm.

Theorem II.14.1

Let $(\mathfrak{A}_i, G_i, \alpha_i)$, $i = 1, 2$, be C^*-systems with G_i compact and abelian, and $\mathfrak{B}$ an arbitrary unital C^*-algebra. If the unital *-homomorphisms

$$\pi_i : \mathfrak{A}_i \to \mathfrak{B}, \quad i = 1, 2\,,$$

satisfy the commutation relation

$$\pi_1(a_1)\pi_2(a_2) = u(a_1, a_2)\pi_2(a_2)\pi_1(a_1)$$

for $a_1 \in \mathfrak{A}_1$, $a_2 \in \mathfrak{A}_2$ both homogeneous, then there exists a unique *-homomorphism $\pi\colon \mathfrak{A}_1 \circledcirc_{\max} \mathfrak{A}_2 \to \mathfrak{B}$ such that

$$\pi(a_1 \odot a_2) = \pi_1(a_1)\pi_2(a_2)\,, \quad a_i \in (\mathfrak{A}_i)_o,\ i = 1, 2\,. \tag{II.18}$$

Moreover, $\pi\big(\mathfrak{A}_1 \circledast_{\max} \mathfrak{A}_2\big)$ is the C^*-subalgebra of $\mathfrak{B}$ generated by $\pi_1(\mathfrak{A}_1)$ and $\pi_2(\mathfrak{A}_2)$. Let $G_1 = G = G_2$ and $G \ni g \mapsto \beta_g \in \mathsf{Aut}(\mathfrak{B})$ an action of G on $\mathfrak{B}$ such that

$$\pi_i \circ (\alpha_i)_g = \beta_g \circ \pi_i\,, \quad g \in G,\ i = 1, 2\,,$$

then $\pi \circ (\alpha_1 \circledast_{\max} \alpha_2)_g = \beta_g \circ \pi$, $g \in G$.

Proof.
The map $(a_1, a_2) \in (\mathfrak{A}_1)_o \times (\mathfrak{A}_2)_o \mapsto \pi_1(a_1)\pi_2(a_2) \in \mathfrak{B}$ is bilinear. Therefore, by the universal property of the tensor product $(\mathfrak{A}_1)_o \odot (\mathfrak{A}_2)_o$ (coinciding with $(\mathfrak{A}_1)_o \circledast (\mathfrak{A}_2)_o$ as a linear space), there is a unique linear map $\pi_o : (\mathfrak{A}_1)_o \circledast (\mathfrak{A}_2)_o \to \mathfrak{B}$ such that

$$\pi_o(a_1 \circledast a_2) = \pi_1(a_1)\pi_2(a_2), \quad a_i \in (\mathfrak{A}_i)_o,\ i = 1, 2\,.$$

It is easy to see that π_o is a *-homomorphism.
After taking a faithful representation $(\rho, \mathcal{H})$ of the C^*-algebra $\mathfrak{B}$, we get that $(\rho \circ \pi_o, \mathcal{H})$ is a representation of $(\mathfrak{A}_1)_o \circledast (\mathfrak{A}_2)_o$ which is bounded under the max-norm. Therefore, it extends to a representation $(\widetilde{\pi}, \mathcal{H})$ of $\mathfrak{A}_1 \circledast_{\max} \mathfrak{A}_2$. Thus $\pi : \rho^{-1} \circ \widetilde{\pi}$ satisfies (II.18) and is the *-homomorphism we are searching for. Moreover, $\pi\big(\mathfrak{A}_1 \circledast_{\max} \mathfrak{A}_2\big)$ is the C^*-subalgebra of $\mathfrak{B}$ generated by $\pi_1(\mathfrak{A}_1)$ and $\pi_2(\mathfrak{A}_2)$.

As concerns the last assertion, for $c = \sum_{i=1}^n x_i \circledast y_i \in (\mathfrak{A}_1)_o \circledast (\mathfrak{A}_2)_o$, we get

$$\begin{aligned} \pi\big((\alpha_1)_g \times (\alpha_2)_g(c)\big) &= \sum_{i=1}^n \pi\big((\alpha_1)_g(x_i) \circledast (\alpha_2)_g(y_i)\big) \\ =&\sum_{i=1}^n \pi_1((\alpha_1)_g(x_i))\pi_2((\alpha_2)_g(y_i)) = \sum_{i=1}^n \beta_g(\pi_1(x_i))\beta_g(\pi_2(y_i)) \\ =&\beta_g\Big(\sum_{i=1}^n \pi_1(x_i)\pi_2(y_i)\Big) = \beta_g(\pi(c))\,. \end{aligned}$$

Since the max-norm is compatible, we now notice that all maps $(\alpha_1)_g \times (\alpha_2)_g$, $g \in G$ defined on the algebraic part $(\mathfrak{A}_1)_o \circledast (\mathfrak{A}_2)_o$ extend to *-automorphisms of $\mathfrak{A}_1 \circledast_{\max} \mathfrak{A}_2$, and the assertion follows. $\square$

Now we go ahead with some characterizations of the max-norm which might be interesting in themselves.

Proposition II.14.2
For each fixed $x \in \mathfrak{A}_o \circledast \mathfrak{B}_o$, the five subsets of $[0, +\infty)$

- $\mathcal{I}_I(x) := \{\|x\| : \|\ \|\ C^*\text{-norm on } \mathfrak{A}_o \circledast \mathfrak{B}_o\}$,
- $\mathcal{I}_{II}(x) := \{\|\pi(x)\| : \pi \in \mathsf{Rep}(\mathfrak{A}_o \circledast \mathfrak{B}_o)\}$,
- $\mathcal{I}_{III}(x) := \{\|\pi(x)\| : \pi \in \mathsf{Rep}(\mathfrak{A}_o \circledast \mathfrak{B}_o) \text{ cyclic}\}$,
- $\mathcal{I}_{IV}(x) := \{\|\pi_f(x)\| : f \in \mathcal{S}(\mathfrak{A}_o \circledast \mathfrak{B}_o)\}$,
- $\mathcal{I}_V(x) := \{\|\pi_f(x)\| : f \in \mathcal{S}_{G\times H}(\mathfrak{A}_o \circledast \mathfrak{B}_o)\}$

share the same (finite) least upper bound, which necessarily coincides with $\|x\|_{\max}$.

Proof.
Clearly $\sup\mathcal{I}_{III}(x) \leq \sup\mathcal{I}_{II}(x)$. Since a cyclic representation with cyclic unit vector ξ is the GNS of the vector state ω_ξ, $\mathcal{I}_{III}(x) = \mathcal{I}_{IV}(x)$. On the other hand, by Zorn Lemma, any representation is a direct sum of cyclic ones, and then we argue that

$$\|x\|_{\max} = \sup\mathcal{I}_{II}(x) = \sup\mathcal{I}_{III}(x) = \sup\mathcal{I}_{IV}(x)\,, \quad x \in \mathfrak{A}_o \textcircled{u} \mathfrak{B}_o\,.$$

Theorem II.10.5 tells us that

$$\sup\mathcal{I}_{I}(x) = \sup\mathcal{I}_{IV}(x)\,, \quad x \in \mathfrak{A}_o \textcircled{u} \mathfrak{B}_o\,.$$

Now, the max-norm is compatible and the invariant states $\mathcal{S}_{G\times H}(\mathfrak{A} \textcircled{u}_{\max} \mathfrak{B})$ under the action $\alpha \textcircled{u}_{\max} \beta$ of $G \times H$ on $\mathfrak{A} \textcircled{u}_{\max} \mathfrak{B}$ separate the points of $\mathfrak{A} \textcircled{u}_{\max} \mathfrak{B}$ and, a fortiori, the points of the algebraic part. Therefore, $\mathcal{S}_{G\times H}(\mathfrak{A} \textcircled{u}_{\max} \mathfrak{B})|_{\mathfrak{A}_o \textcircled{u} \mathfrak{B}_o} = \mathcal{S}_{G\times H}(\mathfrak{A}_o \textcircled{u} \mathfrak{B}_o)$ generates the same norm as the max-norm. □

Remark II.14.3
The results of the previous proposition can be summarized as

$$\begin{aligned}\|\ \|_{\max} &= \sup_{\substack{\pi\in\mathsf{Rep}(\mathfrak{A}_o \textcircled{u} \mathfrak{B}_o)\\ \text{(cyclic)}}} \|\pi(\cdot)\| = \sup_{f\in\mathcal{S}(\mathfrak{A}_o \textcircled{u} \mathfrak{B}_o)} \|\pi_f(\cdot)\| \\ &= \sup_{f\in\mathcal{S}_{G\times H}(\mathfrak{A}_o \textcircled{u} \mathfrak{B}_o)} \|\pi_f(\cdot)\| = \sup_{\gamma\, C^*\text{-norm}} \|\ \|_\gamma\,.\end{aligned}$$

II.15. Characterizations of the min-norm

The main aim of the present section is to show the spatiality of the min-norm, well known for the usual tensor product. In this case, the proof of such a fundamental property is more involved, as expected. In our framework based on the C^*-systems $(\mathfrak{A}, G, \alpha)$ and $(\mathfrak{B}, H, \beta)$, we refer to the definition of twisted tensor product of representations in Section II.8, as well as the covariant construction of a representation in (II.3).
The following result, connected to the spatiality of the min-norm, is of relevant interest, due to its generality.

Proposition II.15.1
Let $(\mathfrak{A}, G, \alpha)$, $(\mathfrak{B}, H, \beta)$ be C^*-systems with G, H compact abelian groups, and $\pi \in \mathsf{Rep}(\mathfrak{A})$, $\rho \in \mathsf{Rep}(\mathfrak{B})$ faithful representations. Then, for each $x \in \mathfrak{A}_o \textcircled{u} \mathfrak{B}_o$,

$$\|(\pi^G_{\lambda^\pi_G} \textcircled{u} \rho^H)(x)\|_{\mathcal{B}(\mathcal{H}_{\pi^G}\otimes\mathcal{H}_{\rho^H})} = \|x\|_{\min} = \|(\pi^G \textcircled{u}\, {}_{\lambda^\rho_H}\rho^H)(x)\|_{\mathcal{B}(\mathcal{H}_{\pi^G}\otimes\mathcal{H}_{\rho^H})}\,.$$

As a consequence,

$$\overline{(\pi^G_{\lambda^\pi_G} \textcircled{u} \rho^H)(\mathfrak{A}_o \textcircled{u} \mathfrak{B}_o)} \cong \mathfrak{A} \textcircled{u}_{\min} \mathfrak{B} \cong \overline{(\pi^G \textcircled{u}\, {}_{\lambda^\rho_H}\rho^H)(\mathfrak{A}_o \textcircled{u} \mathfrak{B}_o)}\,.$$

Proof.
For fixed $A \in \mathcal{B}(\mathcal{H}_{\pi^G})$ and $B \in \mathcal{B}(\mathcal{H}_{\rho^H})$, define

$$E(A) := \int_G \lambda^\pi_G(g^{-1})A\lambda^\pi_G(g)\mathrm{d}g\,, \quad F(B) := \int_H \lambda^\rho_H(h^{-1})B\lambda^\rho_H(h)\mathrm{d}h\,,$$

where the integrals are meant in the weak operator topology. First note that $E(A)$ $(F(B))$ is G (H) invariant. If A (B) is positive and/or trace-class, so is $E(A)$ $(F(B))$. Define now $\mathcal{S}_G(\pi^G) := \mathcal{F}(\pi^G) \cap \mathcal{S}_G(\mathfrak{A})$ and $\mathcal{S}_H(\rho^H) := \mathcal{F}(\rho^H) \cap \mathcal{S}_H(\mathfrak{B})$ (cf. Section II.4).

If $\omega \in \mathcal{S}_G(\mathfrak{A})$ and $\varphi \in \mathcal{S}_H(\mathfrak{B})$, by faithfulness of π^G and ρ^H there exist two nets of states $\widetilde{\omega}_\iota := \mathrm{tr}_{\mathcal{H}_{\pi^G}}(\widetilde{R}_\iota \pi^G(\cdot)) \in \mathcal{F}(\pi^G)$ and $\widetilde{\varphi}_\kappa := \mathrm{tr}_{\mathcal{H}_{\rho^H}}(\widetilde{S}_\kappa \rho^H(\cdot)) \in \mathcal{F}(\rho^H)$ weakly-* converging to ω and φ respectively, with $\widetilde{R}_\iota \in \mathcal{B}_1(\mathcal{H}_{\pi^G})$ and $\widetilde{S}_\kappa \in \mathcal{B}_1(\mathcal{H}_{\rho^H})$ positive unit-trace operators (see also [104], Theorem IV.4.9 (iii), p. 208). By Lebesgue Dominated Convergence Theorem, setting $R_\iota := E(\widetilde{R}_\iota)$ and $S_\kappa := F(\widetilde{S}_\kappa)$, the nets $(\omega_\iota)_\iota \subset \mathcal{S}_G(\pi^G)$ and $(\varphi_\kappa)_\kappa \subset \mathcal{S}_H(\rho^H)$, where $\omega_\iota := \mathrm{tr}_{\mathcal{H}_{\pi^G}}(R_\iota \pi^G(\cdot))$, $\varphi_\kappa := \mathrm{tr}_{\mathcal{H}_{\rho^H}}(S_\kappa \rho^H(\cdot))$, still converge to ω and φ, respectively. Notice that for each pair (ι, κ), $R_\iota \otimes S_\kappa$ is a positive unit-trace operator, acting on $\mathcal{H}_{\pi^G} \otimes \mathcal{H}_{\rho^H}$ and reproducing $\omega_\iota \times \varphi_\kappa$ as a state on the whole $\mathcal{B}(\mathcal{H}_{\pi^G} \otimes \mathcal{H}_{\rho^H})$. Furthermore, the product state $\omega \times \varphi$ is seen at once as the weak-* limit of the net $(\omega_\iota \times \varphi_\kappa)_{\iota,\kappa}$ (see the proof of Proposition 4.5 of [19], p. 13, for further details).
Lastly, since the states of the form $\mathrm{tr}_{\mathcal{H}_{\pi^G} \otimes \mathcal{H}_{\rho^H}}((R \otimes S)\,\cdot)$ ($R \in \mathcal{B}_1(\mathcal{H}_{\pi^G})$, $S \in \mathcal{B}_1(\mathcal{H}_{\rho^H})$ positive, unit-trace and invariant) separate the points of $\mathcal{B}(\mathcal{H}_{\pi^G} \otimes \mathcal{H}_{\rho^H})$, and in particular of both $(\pi^G_{\lambda^\pi_G} ⓤ \rho^H)(\mathfrak{A}_o ⓤ \mathfrak{B}_o)$ and $(\pi^G ⓤ {}_{\lambda^\rho_H}\rho^H)(\mathfrak{A}_o ⓤ \mathfrak{B}_o)$, we can conclude the proof. For each $x \in \mathfrak{A}_o ⓤ \mathfrak{B}_o$,

$$\|x\|_{\min} = \sup_{\substack{\omega \in \mathcal{S}_G(\mathfrak{A}) \\ \varphi \in \mathcal{S}_H(\mathfrak{B})}} \|(\pi_\omega ⓤ \pi_\varphi)(x)\| =$$
$$= \sup_{\substack{\omega \in \mathcal{S}_G(\pi^G) \\ \varphi \in \mathcal{S}_H(\rho^H)}} \|(\pi_\omega ⓤ \pi_\varphi)(x)\| = \begin{cases} \|(\pi^G_{\lambda^\pi_G} ⓤ \rho^H)(x)\|\,, \\ \|(\pi^G ⓤ {}_{\lambda^\rho_H}\rho^H)(x)\| \end{cases}$$

and the proof is accomplished. □

Remark II.15.2
Notice that the proof of the proposition above can be replicated for the twisted tensor product $\pi_{U^\pi} ⓤ \rho^H$ or $\widetilde{\pi}^G ⓤ {}_{U^{\widetilde{\rho}}}\widetilde{\rho}$ for $\pi \in \mathsf{Cov}(\mathfrak{A})$, $\rho \in \mathsf{Rep}(\mathfrak{B})$ or $\widetilde{\pi} \in \mathsf{Rep}(\mathfrak{A})$, $\widetilde{\rho} \in \mathsf{Cov}(\mathfrak{B})$ respectively, where all the involved representations are faithful. We then can write

$$\overline{(\pi_{U^\pi} ⓤ \rho^H)(\mathfrak{A}_o ⓤ \mathfrak{B}_o)} \cong \mathfrak{A} ⓤ_{\min} \mathfrak{B} \cong \overline{(\widetilde{\pi}^G ⓤ {}_{U^{\widetilde{\rho}}}\widetilde{\rho})(\mathfrak{A}_o ⓤ \mathfrak{B}_o)}\,.$$

For a C^*-system (with G compact and, in our situation, also abelian) $(\mathfrak{A}, G, \alpha)$, notice that $\bigoplus_{\omega \in \mathcal{S}_G(\mathfrak{A})} \pi_\omega$ and $\Big(\bigoplus_{\omega \in \mathcal{S}(\mathfrak{A})} \pi_\omega\Big)^G$ acting on $\bigoplus_{\omega \in \mathcal{S}_G(\mathfrak{A})} \mathcal{H}_\omega$ and $\bigoplus_{\omega \in \mathcal{S}(\mathfrak{A})} L^2(G, \mathrm{d}g; \mathcal{H}_\omega)$ respectively, are two useful covariant, faithful representations. Apart from faithfulness, this suggests that the covariance property is needed only for the construction of the twisted product of such representations, as it is explained by the following result, which also includes the spatiality of the min-norm.

Proposition II.15.3
For each fixed $x \in \mathfrak{A}_o ⓤ \mathfrak{B}_o$, all subsets of $[0, +\infty)$

- $\mathcal{I}^{\mathrm{L}}_{I}(x) := \{\|(\pi_U ⓤ \rho)(x)\| \colon (\pi, U) \in \mathsf{Cov}(\mathfrak{A}, G, \alpha), \rho \in \mathsf{Rep}(\mathfrak{B})\}$,

  $\mathcal{I}^{\mathrm{R}}_{I}(x) := \{\|(\pi ⓤ {}_V\rho)(x)\| \colon \pi \in \mathsf{Rep}(\mathfrak{A}), (\rho, V) \in \mathsf{Cov}(\mathfrak{B}, H, \beta)\}$,
- $\mathcal{I}^{\mathrm{L}}_{II}(x) := \{\|\pi_{\omega \times \varphi}(x)\| \colon \omega \in \mathcal{S}_G(\mathfrak{A}), \varphi \in \mathcal{S}(\mathfrak{B})\}$,

  $\mathcal{I}^{\mathrm{R}}_{II}(x) := \{\|\pi_{\omega \times \varphi}(x)\| \colon \omega \in \mathcal{S}(\mathfrak{A}), \varphi \in \mathcal{S}_H(\mathfrak{B})\}$,
- $\mathcal{I}^{\mathrm{L}}_{III}(x) := \big\{\big\|\big((\pi_\omega)_{U_\omega} ⓤ \pi_\varphi\big)(x)\big\| \colon \omega \in \mathcal{S}_G(\mathfrak{A}), \varphi \in \mathcal{S}(\mathfrak{B})\big\}$,

  $\mathcal{I}^{\mathrm{R}}_{III}(x) := \big\{\big\|\big(\pi_\omega ⓤ {}_{V_\varphi}\pi_\varphi\big)(x)\big\| \colon \omega \in \mathcal{S}(\mathfrak{A}), \varphi \in \mathcal{S}_H(\mathfrak{B})\big\}$

share the same (finite) least upper bound, which coincides with

$$\|(\pi_{U^\pi} \circledast \rho)(x)\| \ \text{ and } \|(\widetilde{\pi} \circledast {}_{V^{\widetilde{\rho}}}\widetilde{\rho})(x)\|\,,$$

for any pair of *faithful* representations $\pi \in \mathsf{Cov}(\mathfrak{A})$, $\rho \in \mathsf{Rep}(\mathfrak{B})$ and $\widetilde{\pi} \in \mathsf{Rep}(\mathfrak{A})$, $\widetilde{\rho} \in \mathsf{Cov}(\mathfrak{B})$.

Proof.
We show the equivalence among the left-handed assertions (the equivalence between the right-handed assertions being analogous). Once done it, the equivalence between the left and right-handed assertions is guaranteed, as

$$\begin{aligned}\sup \mathcal{I}^{\mathrm{L}}_{III}(x) &= \sup_{\substack{\omega\in\mathcal{S}_G(\mathfrak{A})\\ \varphi\in\mathcal{S}(\mathfrak{B})}} \|((\pi_\omega)_{U_\omega} \circledast \pi_\varphi)(x)\| = \sup_{\substack{\omega\in\mathcal{S}_G(\mathfrak{A})\\ \varphi\in\mathcal{S}_H(\mathfrak{B})}} \|((\pi_\omega)_{U_\omega} \circledast \pi_\varphi)(x)\| \\ &= \sup_{\substack{\omega\in\mathcal{S}_G(\mathfrak{A})\\ \varphi\in\mathcal{S}_H(\mathfrak{B})}} \|(\pi_\omega \circledast {}_{V_\varphi}(\pi_\varphi))(x)\| = \sup_{\substack{\omega\in\mathcal{S}(\mathfrak{A})\\ \varphi\in\mathcal{S}_H(\mathfrak{B})}} \|(\pi_\omega \circledast {}_{V_\varphi}(\pi_\varphi))(x)\| = \sup \mathcal{I}^{\mathrm{R}}_{III}(x)\,.\end{aligned}$$

Since the product functional of two states $\omega \in \mathcal{S}(\mathfrak{A})$, $\varphi \in \mathcal{S}(\mathfrak{B})$ is positive whenever at least one is invariant (cf. Proposition II.9.1), say ω, it is a matter of routine to check that $(\pi_\omega)_{U_\omega} \circledast \pi_\varphi \sim \pi_{\omega\times\varphi}$, hence $\mathcal{I}^{\mathrm{L}}_{II}(x) = \mathcal{I}^{\mathrm{L}}_{III}(x)$. Clearly, $\sup \mathcal{I}^{\mathrm{L}}_{III}(x) \leq \sup \mathcal{I}^{\mathrm{L}}_{I}(x)$, thus we are just left to show that $\sup \mathcal{I}^{\mathrm{L}}_{I}(x) \leq \sup \mathcal{I}^{\mathrm{L}}_{II}(x)$.
Firstly, we shall prove that if (π, U) is a covariant, not necessarily faithful, representation of $(\mathfrak{A}, G, \alpha)$ and ρ_1, ρ_2 are two faithful representations of $\mathfrak{B}$ then, on $\mathfrak{A}_o \circledast \mathfrak{B}_o$,

$$\|(\pi_U \circledast \rho_1)(\cdot)\|_{\mathcal{B}(\mathcal{H}_\pi\otimes\mathcal{H}_{\rho_1})} = \|(\pi_U \circledast \rho_2)(\cdot)\|_{\mathcal{B}(\mathcal{H}_\pi\otimes\mathcal{H}_{\rho_2})}\,. \tag{II.19}$$

We are going to use a compression technique via orthogonal projections, as in Proposition 3.3.11 of [88] (p. 75).
Let $(P_\lambda)_{\lambda\in\Lambda} \subset \mathcal{B}(\mathcal{H}_\pi)$ be an increasing net of finite-rank projections, with $n_\lambda := \mathrm{rk}(P_\lambda) \in \mathbb{N}$, such that $P_\lambda \uparrow I_{\mathcal{H}_\pi}$. From now on, let $\lambda \in \Lambda$ be fixed. Then, the mapping

$$X \mapsto \|X\|_{\lambda,\rho} := \|(P_\lambda \otimes I_{\mathcal{H}_\rho})X(P_\lambda \otimes I_{\mathcal{H}_\rho})\|_{\mathcal{B}(\mathcal{H}_\pi\otimes\mathcal{H}_\rho)}$$

defines a C^*-seminorm on $\mathcal{B}(\mathcal{H}_\pi \otimes \mathcal{H}_\rho)$, which is a C^*-norm (actually the unique possible one), when restricted to $(P_\lambda \otimes I_{\mathcal{H}_\rho})\mathcal{B}(\mathcal{H}_\pi \otimes \mathcal{H}_\rho)(P_\lambda \otimes I_{\mathcal{H}_\rho}) \cong M_{n_\lambda}(\mathbb{C}) \otimes \mathcal{B}(\mathcal{H}_\rho)$.
By applying the mapping above to two faithful representations ρ_j of $\mathfrak{B}$ and to operators $X_j := (\pi_U \circledast \rho_j)\Big(\sum_{i=1}^n a_i \circledast b_i\Big)$, $j = 1, 2$, with the b_i's homogeneous, we obtain

$$\begin{aligned}\Big\|(\pi_U \circledast \rho_1)\Big(\sum_{i=1}^n a_i \circledast b_i\Big)\Big\|_{\mathcal{B}(\mathcal{H}_\pi\otimes\mathcal{H}_{\rho_1})} &= \sup_{\lambda\in\Lambda}\Big\|(\pi_U \circledast \rho_1)\Big(\sum_{i=1}^n a_i \circledast b_i\Big)\Big\|_{\lambda,\rho_1} \\ &= \sup_{\lambda\in\Lambda}\Big\|\sum_{i=1}^n P_\lambda\pi(a_i)U(g_{\partial b_i})P_\lambda \otimes \rho_1(b_i)\Big\|_{\mathcal{B}(\mathcal{H}_\pi\otimes\mathcal{H}_{\rho_1})} = \\ &= \sup_{\lambda\in\Lambda}\Big\|\sum_{i=1}^n P_\lambda\pi(a_i)U(g_{\partial b_i})P_\lambda \otimes \rho_2(b_i)\Big\|_{\mathcal{B}(\mathcal{H}_\pi\otimes\mathcal{H}_{\rho_2})} = \\ = \sup_{\lambda\in\Lambda}\Big\|(\pi_U \circledast \rho_2)\Big(\sum_{i=1}^n a_i \circledast b_i\Big)\Big\|_{\lambda,\rho_2} &= \Big\|(\pi_U \circledast \rho_2)\Big(\sum_{i=1}^n a_i \circledast b_i\Big)\Big\|_{\mathcal{B}(\mathcal{H}_\pi\otimes\mathcal{H}_{\rho_2})}\,.\end{aligned}$$

Applying (II.19) to the special case where $(\pi, U) \in \mathsf{Cov}(\mathfrak{A}, G, \alpha)$ and $\rho \in \mathsf{Rep}(\mathfrak{B})$ are faithful, $\rho_1 := \rho$ and $\rho_2 := \rho^H$,

$$\|(\pi_U \circledast \rho)(\cdot)\|_{\mathcal{B}(\mathcal{H}_\pi\otimes\mathcal{H}_\rho)} = \|(\pi_U \circledast \rho^H)(\cdot)\|_{\mathcal{B}(\mathcal{H}_\pi\otimes\mathcal{H}_{\rho^H})} = \|\cdot\|_{\min}\,,$$

where the second equality is due to Remark II.15.2.
This allows us to conclude the proof, by showing that $\sup \mathcal{I}_I(x) \le \sup \mathcal{I}_{II}(x)$. Indeed, consider any $(\pi, U) \in \mathsf{Cov}(\mathfrak{A}, G, \alpha)$ and $\rho \in \mathsf{Rep}(\mathfrak{B})$. If we take a pair of faithful representations $\widetilde{\pi} \in \mathsf{Rep}(\mathfrak{A})$ and $\widetilde{\rho} \in \mathsf{Rep}(\mathfrak{B})$,

$$\begin{aligned}
&\|(\pi_U \mathbin{\textcircled{w}} \rho)(\cdot)\| \\
&\le \max\left\{\|(\pi_U \mathbin{\textcircled{w}} \rho)(\cdot)\|, \|(\pi_U \mathbin{\textcircled{w}} \widetilde{\rho}^H)(\cdot)\|, \|(\widetilde{\pi}^G_{\lambda^{\widetilde{\pi}}_G} \mathbin{\textcircled{w}} \rho)(\cdot)\|, \|(\widetilde{\pi}^G_{\lambda^{\widetilde{\pi}}_G} \mathbin{\textcircled{w}} \widetilde{\rho}^H)(\cdot)\|\right\} \\
&= \left\|\Big((\pi_U \mathbin{\textcircled{w}} \rho) \oplus (\pi_U \mathbin{\textcircled{w}} \widetilde{\rho}^H) \oplus (\widetilde{\pi}^G_{\lambda^{\widetilde{\pi}}_G} \mathbin{\textcircled{w}} \rho) \oplus (\widetilde{\pi}^G_{\lambda^{\widetilde{\pi}}_G} \mathbin{\textcircled{w}} \widetilde{\rho}^H)\Big)(\cdot)\right\| \\
&\overset{(\star)}{=} \|(\pi \oplus \widetilde{\pi}^G)_{U \oplus \lambda^{\widetilde{\pi}}_G} \mathbin{\textcircled{w}} (\rho \oplus \widetilde{\rho}^H)(\cdot)\| = \|\cdot\|_{\min} = \sup \mathcal{I}_{II}(x)\,,
\end{aligned}$$

since $(\pi \oplus \widetilde{\pi}^G, U \oplus \lambda^{\widetilde{\pi}}_G) \in \mathsf{Cov}(\mathfrak{A}, G, \alpha)$ and $\rho \oplus \widetilde{\rho}^H \in \mathsf{Rep}(\mathfrak{B})$ are both faithful. Equality $(\star)$ is justified by a direct computation: with $a_i \in \mathfrak{A}_o$ and homogeneous $b_i \in \mathfrak{B}_o$,

$$\begin{aligned}
&\left\|(\pi \oplus \widetilde{\pi}^G)_{U \oplus \lambda^{\widetilde{\pi}}_G} \mathbin{\textcircled{w}} (\rho \oplus \widetilde{\rho}^H)\Big(\sum_{i=1}^n a_i \mathbin{\textcircled{w}} b_i\Big)\right\| \\
&\qquad = \left\|\sum_{i=1}^n \big(\pi(a_i)U(g_{\partial b_i}) \oplus \widetilde{\pi}^G(a_i)\lambda^{\widetilde{\pi}}_G(g_{\partial b_i})\big) \otimes \big(\rho(b_i) \oplus \widetilde{\rho}^H(b_i)\big)\right\| \\
&\qquad = \left\|\Big((\pi_U \mathbin{\textcircled{w}} \rho) \oplus (\pi_U \mathbin{\textcircled{w}} \widetilde{\rho}^H) \oplus (\widetilde{\pi}^G_{\lambda^{\widetilde{\pi}}_G} \mathbin{\textcircled{w}} \rho) \oplus (\widetilde{\pi}^G_{\lambda^{\widetilde{\pi}}_G} \mathbin{\textcircled{w}} \widetilde{\rho}^H)\Big)\Big(\sum_{i=1}^n a_i \mathbin{\textcircled{w}} b_i\Big)\right\|.
\end{aligned}$$

By passing to the least upper bound on every $(\pi, U) \in \mathsf{Cov}(\mathfrak{A}, G, \alpha)$ and $\rho \in \mathsf{Rep}(\mathfrak{B})$, we get $\sup \mathcal{I}_I(x) \le \sup \mathcal{I}_{II}(x)$. □

An immediate consequence of the above result is the following

Corollary II.15.4
Every product state in $\big(\mathcal{S}_G(\mathfrak{A}) \times \mathcal{S}(\mathfrak{B})\big) \cup \big(\mathcal{S}(\mathfrak{A}) \times \mathcal{S}_H(\mathfrak{B})\big)$ on $\mathfrak{A}_o \mathbin{\textcircled{w}} \mathfrak{B}_o$ extends to a state on the whole $\mathfrak{A} \mathbin{\textcircled{w}}_{\min} \mathfrak{B}$.

II.16. Isometric extensions

This section is devoted to the isometric extensions of

- the right and left factoring-out mappings R and L in (II.1) to both the minimal and maximal twisted C^*-completions, using (II.8), (II.9), (II.10), (II.11) and (II.12) to define the appropriate actions and bicharacters
- the algebraic isomorphism Proposition II.7.4 w.r.t. the minimal C^*-completion

Let us start with R and L.

Proposition II.16.1
The factoring-out mappings R and L in (II.1) extend to *-isomorphisms of the following C^*-algebras:

$$\begin{aligned}
R_\gamma &\colon (\mathfrak{A}_1 \mathbin{\textcircled{u_1}}_\gamma \mathfrak{B}) \oplus (\mathfrak{A}_2 \mathbin{\textcircled{u_2}}_\gamma \mathfrak{B}) \to (\mathfrak{A}_1 \oplus \mathfrak{A}_2) \mathbin{\textcircled{w}}_\gamma \mathfrak{B}\,, \\
L_\gamma &\colon (\mathfrak{A} \mathbin{\textcircled{u_1}}_\gamma \mathfrak{B}_1) \oplus (\mathfrak{A} \mathbin{\textcircled{u_2}}_\gamma \mathfrak{B}_2) \to \mathfrak{A} \mathbin{\textcircled{w}}_\gamma (\mathfrak{B}_1 \oplus \mathfrak{B}_2)\,,
\end{aligned}$$

where $\gamma \in \{\|\ \|_{\min}, \|\ \|_{\max}\}$.

Proof.
Let us prove these facts for R, bearing in mind that a similar proof holds for L too. We start with $\gamma = \| \; \|_{\min}$, by exploiting Proposition II.15.3.
Let $\pi_k \in \mathsf{Rep}(\mathfrak{A}_k)$, $\rho \in \mathsf{Rep}(\mathfrak{B})$ be three faithful representations on $\mathcal{H}_k$ $(k = 1, 2)$ and $\mathcal{H}_\rho$, respectively. Therefore, the triplet

$$(\pi_1^{G_1} \oplus \pi_2^{G_2}, \lambda_{G_1}^{\pi_1} \oplus \lambda_{G_2}^{\pi_2}, \mathcal{H}_{\pi_1^{G_1}} \oplus \mathcal{H}_{\pi_2^{G_2}})$$

defines a faithful covariant representation of $(\mathfrak{A}_1 \oplus \mathfrak{A}_2, G_1 \times G_2, \alpha_1 \oplus \alpha_2)$. By Proposition II.15.3, for $a_{k,i} \in (\mathfrak{A}_k)_o$ and homogeneous $b_i, B_j \in \mathfrak{B}_o$, where $i = 1, \dots, m$, $j = 1, \dots, n$, we have

$$\begin{aligned}
\Big\| R\Big(\sum_{i=1}^m a_{1,i} \textcircled{u} b_i, \sum_{j=1}^n a_{2,j} \textcircled{u} B_j \Big) \Big\|_{\min} &= \Big\| \sum_{i=1}^m (a_{1,i}, 0) \textcircled{u} b_i + \sum_{j=1}^n (0, a_{2,j}) \textcircled{u} B_j \Big\|_{\min} \\
&= \Big\| \sum_{i=1}^m (\pi^{G_1}(a_{1,i}) \oplus 0)(\lambda_{G_1}^\pi \oplus \lambda_{G_2}^\pi)(\mathbf{g}_{\partial b_i}) \otimes \rho(b_i) \\
&\quad + \sum_{j=1}^n (0 \oplus \pi^{G_2}(a_{2,j}))(\lambda_{G_1}^\pi \oplus \lambda_{G_2}^\pi)(\mathbf{g}_{\partial B_j}) \otimes \rho(B_j) \Big\|_{\mathcal{B}((\mathcal{H}_{\pi^{G_1}} \oplus \mathcal{H}_{\pi^{G_2}}) \otimes \mathcal{H}_\rho)}
\end{aligned}$$

where, for each homogeneous $b \in \mathfrak{B}_o$, $\mathbf{g}_{\partial b} \in G_1 \times G_2$ is uniquely determined by the condition $\mathrm{ev}_{\mathbf{g}_{\partial b}}(\sigma_1, \sigma_2) = \overline{u((\sigma_1, \sigma_2), \partial b)}$, $(\sigma_1, \sigma_2) \in \widehat{G_1} \times \widehat{G_2}$. Notice also that there exists a unique pair $(g_{1,\partial b}, g_{2,\partial b}) \in G_1 \times G_2$ such that $\mathrm{ev}_{g_{1,\partial b}}(\sigma_1) = \overline{u_1(\sigma_1, \partial b)}$ for every $\sigma_1 \in \widehat{G_1}$ and $\mathrm{ev}_{g_{2,\partial b}}(\sigma_2) = \overline{u_2(\sigma_2, \partial b)}$ for every $\sigma_2 \in \widehat{G_2}$. It straightforwardly results that $\mathbf{g}_{\partial b} = (g_{1,\partial b}, g_{2,\partial b}) \in G_1 \times G_2$ for every homogeneous $b \in \mathfrak{B}_o$.
Collecting all together and setting $\mathcal{H} := (\mathcal{H}_{\pi_1^{G_1}} \otimes \mathcal{H}_\rho) \oplus (\mathcal{H}_{\pi_2^{G_2}} \otimes \mathcal{H}_\rho)$, we get

$$\begin{aligned}
&\Big\| R\Big(\sum_{i=1}^m a_{1,i} \textcircled{u} b_i, \sum_{j=1}^n a_{2,j} \textcircled{u} B_j \Big) \Big\|_{\min} = \\
&= \Big\| \Big(\sum_{i=1}^m \pi^{G_1}(a_{1,i}) \lambda_{G_1}^\pi(g_{1,\partial b_i}) \otimes \rho(b_i) \Big) \oplus \Big(\sum_{j=1}^n \pi^{G_2}(a_{2,j}) \lambda_{G_2}^\pi(g_{2,\partial B_j}) \otimes \rho(B_j) \Big) \Big\|_{\mathcal{B}(\mathcal{H})} = \\
&= \Big\| \sum_{i=1}^m \pi^{G_1}(a_{1,i}) \lambda_{G_1}^\pi(g_{1,\partial b_i}) \otimes \rho(b_i) \Big\|_{\mathcal{B}(\mathcal{H}_{\pi_1^{G_1}} \otimes \mathcal{H}_\rho)} \\
&\quad \bigvee \Big\| \sum_{j=1}^n \pi^{G_2}(a_{2,j}) \lambda_{G_2}^\pi(g_{2,\partial B_j}) \otimes \rho(B_j) \Big\|_{\mathcal{B}(\mathcal{H}_{\pi_2^{G_2}} \otimes \mathcal{H}_\rho)} = \\
&= \Big\| \sum_{i=1}^m a_{1,i} \textcircled{u} b_i \Big\|_{\min} \bigvee \Big\| \sum_{j=1}^n a_{2,j} \textcircled{u} B_j \Big\|_{\min}
\end{aligned}$$

that is, R can be isometrically extended to a *-isomorphism of C^*-algebras

$$R_{\min} \colon (\mathfrak{A}_1 \textcircled{\scriptsize u_1}_{\min} \mathfrak{B}) \oplus (\mathfrak{A}_2 \textcircled{\scriptsize u_2}_{\min} \mathfrak{B}) \to (\mathfrak{A}_1 \oplus \mathfrak{A}_2) \textcircled{u}_{\min} \mathfrak{B} \,.$$

As concerns $\gamma = \| \; \|_{\max}$, we will make use of Theorem II.14.1. The *-homomorphisms

$$R_{\mathfrak{A}_1}(a_1) := R(a_1 \textcircled{u} \mathbb{1}_{\mathfrak{B}}, 0) = (a_1, 0) \textcircled{u} \mathbb{1}_{\mathfrak{B}}, \quad a_1 \in (\mathfrak{A}_1)_o$$

$$R_{1,\mathfrak{B}}(b) := R(\mathbb{1}_{\mathfrak{A}_1} \textcircled{u} b, 0) = (\mathbb{1}_{\mathfrak{A}_1}, 0) \textcircled{u} b, \quad b \in \mathfrak{B}_o$$

satisfy the relation $R_{1,\mathfrak{B}}(b)R_{\mathfrak{A}_1}(a_1) = u_1(a_1,b)R_{\mathfrak{A}_1}(a_1)R_{1,\mathfrak{B}}(b)$ for every homogeneous $a_1 \in (\mathfrak{A}_1)_o$, $b \in \mathfrak{B}_o$. Likewise,

$$R_{\mathfrak{A}_2}(a_2) := R(0, a_2 \textcircled{u} \mathbb{1}_{\mathfrak{B}}) = (0, a_2) \textcircled{u} \mathbb{1}_{\mathfrak{B}}, \quad a_2 \in (\mathfrak{A}_2)_o$$

$$R_{2,\mathfrak{B}}(b) := R(0, \mathbb{1}_{\mathfrak{A}_2} \textcircled{u}\, b) = (0, \mathbb{1}_{\mathfrak{A}_2}) \textcircled{u}\, b, \quad b \in \mathfrak{B}_o$$

satisfy the equality $R_{2,\mathfrak{B}}(b)R_{\mathfrak{A}_2}(a_2) = u_2(a_2,b)R_{\mathfrak{A}_2}(a_2)R_{2,\mathfrak{B}}(b)$ for every homogeneous $a_2 \in (\mathfrak{A}_2)_o$, $b \in \mathfrak{B}_o$.
By the universal property of the maximal C^*-completion, there exist unique *-homomorphisms of C^*-algebras

$$R_{1,\max} \colon \mathfrak{A}_1 \textcircled{\scriptsize u_1}_{\max} \mathfrak{B} \to (\mathfrak{A}_1 \oplus \mathfrak{A}_2) \textcircled{u}_{\max} \mathfrak{B}$$

$$R_{2,\max} \colon \mathfrak{A}_2 \textcircled{\scriptsize u_2}_{\max} \mathfrak{B} \to (\mathfrak{A}_1 \oplus \mathfrak{A}_2) \textcircled{u}_{\max} \mathfrak{B}$$

such that $R_{1,\max}(a_1 \textcircled{u}\, b) = R_{\mathfrak{A}_1}(a_1)R_{1,\mathfrak{B}}(b) = R(a_1 \textcircled{u}\, b, 0)$ and $R_{2,\max}(a_2 \textcircled{u}\, b) = R_{\mathfrak{A}_2}(a_2)R_{1,\mathfrak{B}}(b) = R(0, a_2 \textcircled{u}\, b)$ for every $a_1 \in (\mathfrak{A}_1)_o$, $a_2 \in (\mathfrak{A}_2)_o$, $b \in \mathfrak{B}_o$. It follows that $R_{\max}(x,y) := R_{1,\max}(x) + R_{2,\max}(y)$ for every $x \in \mathfrak{A}_1 \textcircled{\scriptsize u_1}_{\max} \mathfrak{B}$, $y \in \mathfrak{A}_2 \textcircled{\scriptsize u_2}_{\max} \mathfrak{B}$ is a *-homomorphism, the unique contractive extension of R to the maximal C^*-completions. Precisely,

$$\left\| R\left(\sum_{i=1}^m a_{1,i} \textcircled{u}\, b_i, \sum_{j=1}^n a_{2,j} \textcircled{u}\, B_j \right) \right\|_{\max} \leq \left\| \sum_{i=1}^m a_{1,i} \textcircled{u}\, b_i \right\|_{\max} \bigvee \left\| \sum_{j=1}^n a_{2,j} \textcircled{u}\, B_j \right\|_{\max}$$

for $a_{1,i} \in (\mathfrak{A}_1)_o, a_{2,j} \in (\mathfrak{A}_2)_o, b_i, B_j \in \mathfrak{B}_o$. We are left to show the converse inequality

$$\begin{aligned}
&\left\| \sum_{i=1}^m a_{1,i} \textcircled{u}\, b_i \right\|_{\max} \bigvee \left\| \sum_{j=1}^n a_{2,j} \textcircled{u}\, B_j \right\|_{\max} \\
&\qquad = \sup_{\substack{\pi \in \mathsf{Rep}((\mathfrak{A}_1)_o \textcircled{u} \mathfrak{B}_o) \\ \rho \in \mathsf{Rep}((\mathfrak{A}_2)_o \textcircled{u} \mathfrak{B}_o)}} \left\| \pi\left(\sum_{i=1}^m a_{1,i} \textcircled{u}\, b_i \right) \oplus \rho\left(\sum_{j=1}^n a_{2,j} \textcircled{u}\, B_j \right) \right\| \\
&\qquad \leq \sup_{\sigma \in \mathsf{Rep}((\mathfrak{A}_1 \oplus \mathfrak{A}_2)_o \textcircled{u} \mathfrak{B}_o)} \left\| (\sigma \circ R)\left(\sum_{i=1}^m a_{1,i} \textcircled{u}\, b_i \right) + (\sigma \circ R)\left(\sum_{j=1}^n a_{2,j} \textcircled{u}\, B_j \right) \right\| \\
&\qquad = \left\| R\left(\sum_{i=1}^m a_{1,i} \textcircled{u}\, b_i, \sum_{j=1}^n a_{2,j} \textcircled{u}\, B_j \right) \right\|_{\max}
\end{aligned}$$

since for a fixed $\sigma \in \mathsf{Rep}((\mathfrak{A}_1 \oplus \mathfrak{A}_2)_o \textcircled{u} \mathfrak{B}_o)$, $\sigma \circ R|_{(\mathfrak{A}_k)_o \textcircled{u} \mathfrak{B}_o} \in \mathsf{Rep}((\mathfrak{A}_k)_o \textcircled{u} \mathfrak{B}_o)$, $k = 1, 2$ (not necessarily in direct sum). The proof is then accomplished. □

Remark II.16.2
When $G_1 = G_2 =: G$ ($H_1 = H_2 =: H$) and $u_1 = u_2 =: u$, an analogous result can be achieved by using the diagonal action of G (H), i.e.

$$\begin{aligned}
(\mathfrak{A}_1 \textcircled{u}_\gamma \mathfrak{B}) \oplus (\mathfrak{A}_2 \textcircled{u}_\gamma \mathfrak{B}) &\cong (\mathfrak{A}_1 \oplus \mathfrak{A}_2) \textcircled{u}_\gamma \mathfrak{B}\,, \\
(\mathfrak{A} \textcircled{u}_\gamma \mathfrak{B}_1) \oplus (\mathfrak{A} \textcircled{u}_\gamma \mathfrak{B}_2) &\cong \mathfrak{A} \textcircled{u}_\gamma (\mathfrak{B}_1 \oplus \mathfrak{B}_2)
\end{aligned}$$

where $\gamma \in \{\| \ \|_{\min}, \| \ \|_{\max}\}$.

Proof.
By the previous proposition, it suffices to show that $(\mathfrak{A}_1 \oplus \mathfrak{A}_2) \textcircled{u}_\gamma^{(f)} \mathfrak{B} \cong (\mathfrak{A}_1 \oplus \mathfrak{A}_2) \textcircled{u}_\gamma^{(d)} \mathfrak{B}$, where on the left-hand side $\mathfrak{A}_1 \oplus \mathfrak{A}_2$ is acted upon by $G \times G$ (via the full action f) whereas on the right-hand side by G (via the diagonal action d) and $w((\sigma_1, \sigma_2), \tau) := u(\sigma_1\sigma_2, \tau)$ ($\sigma_1, \sigma_2 \in \widehat{G}, \tau \in \widehat{H}$).

Clearly, $(\mathfrak{A}_1 \oplus \mathfrak{A}_2)_o \textcircled{u}^{(f)} \mathfrak{B}_o \cong (\mathfrak{A}_1 \oplus \mathfrak{A}_2)_o \textcircled{u}^{(d)} \mathfrak{B}_o$ as involutive algebras, hence their maximal completions (defined as lower upper bounds on all their respective representations) must coincide. As concerns their minimal completions, notice that the fixed point algebras $(\mathfrak{A}_1 \oplus \mathfrak{A}_2)^{G,d}$ and $(\mathfrak{A}_1 \oplus \mathfrak{A}_2)^{G\times G,f}$ are the same, hence

$$\|\cdot\|_{\min,f} = \sup_{\substack{\omega\in\mathcal{S}_{G\times G}(\mathfrak{A}_1\oplus\mathfrak{A}_2)\\ \varphi\in\mathcal{S}_H(\mathfrak{B})}} \|\pi_{\omega\times\varphi}(\cdot)\| = \sup_{\substack{\omega\in\mathcal{S}_{G}(\mathfrak{A}_1\oplus\mathfrak{A}_2)\\ \varphi\in\mathcal{S}_H(\mathfrak{B})}} \|\pi_{\omega\times\varphi}(\cdot)\| = \|\cdot\|_{\min,d}\,. \qquad \square$$

We now pass to the algebraic isomorphism in Proposition II.7.4.

Proposition II.16.3
Under the notation introduced before Proposition II.7.4, for every C^*-norm γ, $(\mathfrak{A}_L)_{oo} \textcircled{u_{nd}} (\mathfrak{B}_R)_{oo}$ is γ-dense in $(\mathfrak{A}_L)_o \textcircled{u_{nd}} (\mathfrak{B}_R)_o$. In particular,

$$\mathfrak{A}_G \textcircled{u}_{\min} \mathfrak{B}_H \cong \mathfrak{A}_L \textcircled{u_{nd}}_{\min} \mathfrak{B}_R\,.$$

Proof.
Firstly, observe that $(\mathfrak{A}_L)_{oo}$ is dense in $(\mathfrak{A}_L)_o$ and $(\mathfrak{B}_R)_{oo}$ is dense in $(\mathfrak{B}_R)_o$, for $\sum_{i=1}^n a_i \textcircled{u_{nd}} b_i \in (\mathfrak{A}_L)_o \textcircled{u_{nd}} (\mathfrak{B}_R)_o$. For fixed finite families $(a_i)_{i=1}^n \in (\mathfrak{A}_L)_o$, $(b_i)_{i=1}^n \in (\mathfrak{B}_L)_o$ and $\varepsilon > 0$, there exist $(a_{i,\varepsilon})_{i=1}^n \in (\mathfrak{A}_L)_{oo}$, $(b_{i,\varepsilon})_{i=1}^n \in (\mathfrak{B}_L)_{oo}$ s.t. $\|a_i - a_{i,\varepsilon}\|_{\mathfrak{A}}, \|b_i - b_{i,\varepsilon}\|_{\mathfrak{B}} \le \varepsilon$ for every $i = 1, \dots, n$. Since γ is sub-cross (cf. (i') in Remark II.11.4),

$$\left\|\sum_{i=1}^n a_i \textcircled{u_{nd}} b_i - \sum_{i=1}^n a_{\varepsilon,i} \textcircled{u_{nd}} b_{\varepsilon,i}\right\|_\gamma \le \sum_{i=1}^n \Big(\|a_i - a_{\varepsilon,i}\|_{\mathfrak{A}}\|b_i\|_{\mathfrak{B}} + \|a_{\varepsilon,i}\|_{\mathfrak{A}}\|b_i - b_{\varepsilon,i}\|_{\mathfrak{B}}\Big) \le$$
$$\le \varepsilon\left(n\varepsilon + \sum_{i=1}^n (\|a_i\|_{\mathfrak{A}} + \|b_i\|_{\mathfrak{B}})\right).$$

hence $(\mathfrak{A}_L)_{oo} \textcircled{u_{nd}} (\mathfrak{B}_R)_{oo}$ is γ-dense in $(\mathfrak{A}_L)_o \textcircled{u_{nd}} (\mathfrak{B}_R)_o$. On the other hand, the two C^*-norms

$$\|\cdot\|_{\min,(G,H)} = \sup_{\substack{\omega\in\mathcal{S}_G(\mathfrak{A})\\ \varphi\in\mathcal{S}_H(\mathfrak{B})}} \|\pi_{\omega\times\varphi}(\cdot)\|$$

$$\|\cdot\|_{\min,(L,R)} = \sup_{\substack{\omega\in\mathcal{S}_{L^\perp}(\mathfrak{A})\\ \varphi\in\mathcal{S}_{R^\perp}(\mathfrak{B})}} \|\pi_{\omega\times\varphi}(\cdot)\|$$

coincide on $(\mathfrak{A}_L)_o \textcircled{u_{nd}} (\mathfrak{B}_R)_o$. Indeed, evidently $\|\cdot\|_{\min,(G,H)} \le \|\cdot\|_{\min,(L,R)}$. On the other hand, $\|\cdot\|_{\min,(G,H)}$ is a compatible C^*-norm on $(\mathfrak{A}_L)_o \textcircled{u_{nd}} (\mathfrak{B}_R)_o$ therefore by Theorem II.12.3 $\|\cdot\|_{\min,(L,R)} \le \|\cdot\|_{\min,(G,H)}$.
Now, by Proposition II.7.4 $(\mathfrak{A}_G)_o \textcircled{u} (\mathfrak{B}_H)_o \cong (\mathfrak{A}_L)_{oo} \textcircled{u_{nd}} (\mathfrak{B}_R)_{oo}$, hence $(\mathfrak{A}_G)_o \textcircled{u} (\mathfrak{B}_H)_o$ is dense in $(\mathfrak{A}_L)_o \textcircled{u_{nd}} (\mathfrak{B}_R)_o$ w.r.t. $\|\cdot\|_{\min,(G,H)}$. It follows that

$$\mathfrak{A}_G \textcircled{u}_{\min} \mathfrak{B}_H \cong \mathfrak{A}_L \textcircled{u_{nd}}_{\min} \mathfrak{B}_R\,. \qquad \square$$

The *rational* rotation C^*-algebra gives a pedagogical example of the construction above. Let $G = H = \mathbb{T}$ acting on $\mathfrak{A} = \mathfrak{B} = \mathcal{C}(\mathbb{T})$ via the usual rotation $\alpha_z(f) = \beta_z(f) = f(z\,\cdot)$, for each $z \in \mathbb{T}, f \in \mathcal{C}(\mathbb{T})$. Let $u \in \mathsf{S}(\mathbb{Z})$ defined by $u(x,y) := e^{i2\pi\frac{m}{n}xy}$ $(x,y \in \mathbb{Z})$ with $m,n \in \mathbb{N}, \gcd(m,n) = 1$. Then, $L = R = \mathrm{Rad}(u) = n\mathbb{Z}$ so that

- $L^\perp = R^\perp \cong \mathbb{Z}_n \trianglelefteq \mathbb{T}$ and $\widehat{L} = \widehat{R} \cong \mathbb{T}/\mathbb{Z}_n$

- $u_{\mathrm{nd}}\colon \mathbb{Z}_n \times \mathbb{Z}_n \to \mathbb{T}$, $u_{\mathrm{nd}}(x,y) = e^{i\frac{2\pi}{n}mxy}$, $x, y \in \mathbb{Z}_n$
- $(\mathcal{C}(\mathbb{T})_{\mathbb{T}})_o = \dotplus_{k\in\mathbb{Z}} \mathbb{C}z^k$
- $(\mathcal{C}(\mathbb{T})_{n\mathbb{Z}})_o = \mathcal{C}(\mathbb{T})_{n\mathbb{Z}} = \bigoplus_{j=0}^{n-1} \overline{\dotplus_{k\in\mathbb{Z}} \mathbb{C}z^{j+kn}}$ and $(\mathcal{C}(\mathbb{T})_{n\mathbb{Z}})_{oo} = \bigoplus_{j=0}^{n-1} \dotplus_{k\in\mathbb{Z}} \mathbb{C}z^{j+kn}$

and in view of Proposition II.16.3, $\mathcal{C}(\mathbb{T})_{\mathbb{T}} \textcircled{u}_{\min} \mathcal{C}(\mathbb{T})_{\mathbb{T}} \cong \mathcal{C}(\mathbb{T})_{n\mathbb{Z}} \textcircled{u_{nd}}_{\min} \mathcal{C}(\mathbb{T})_{n\mathbb{Z}}$.

II.17. About nuclearity

Recall that a C^*-algebra $\mathfrak{A}$ is *nuclear* if it satisfies one of the following equivalent properties:

- $\|\cdot\|_{\min} = \|\cdot\|_{\max}$ on $\mathfrak{A} \otimes \mathfrak{B}$ for every C^*-algebra $\mathfrak{B}$
- $I_{\mathfrak{A}} \in \mathsf{Aut}(\mathfrak{A})$ approximately factors through full matrix algebras in the strong (i.e. point-norm) topology of $\mathcal{B}(\mathfrak{A})$:

  $$\|(\psi_n \circ \varphi_n)(a) - a\| \xrightarrow{n\uparrow+\infty} 0, \quad a \in \mathfrak{A}$$

  for contractive, c.p. maps $\mathfrak{A} \underset{\psi_n}{\overset{\varphi_n}{\rightleftarrows}} M_{k_n}(\mathbb{C})$
- its enveloping von Neumann algebra $\mathfrak{A}''$ is injective
- $\mathfrak{A}$ is an amenable Banach algebra

Examples of nuclear C^*-algebras are the abelian ones ($\cong \mathcal{C}_o(X)$ for some locally compact, Hausdorff space X), (approximately) finite-dimensional C^*-algebras, type I C^*-algebras (i.e. all their non-degenerate factor representations are irreducible) and group C^*-algebras $C^*_{\mathrm{f}}(G)$ ($\cong C^*_{\mathrm{r}}(G)$) for G amenable, locally compact and Hausdorff.
In this section, we prove that there exists a unique compatible C^*-norm on $\mathfrak{A}_o \textcircled{u} \mathfrak{B}_o$ if and only if either $\mathfrak{A}$ or $\mathfrak{B}$ is a nuclear C^*-algebra, in which case $\mathfrak{A} \textcircled{u}_{\min} \mathfrak{B} \cong \mathfrak{A} \textcircled{u}_{\max} \mathfrak{B}$. This result is directly accomplished by Corollary 7.7 in [70] (p. 314), if we show that our construction of the min and max-norm is equivalent to the one in [57] and in [70], respectively. The approach of the two mentioned papers is very general, as they deal with C^*-quantum groups instead of classical ones. For the convenience of the reader, here we briefly sketch their construction. We are given three data:

(1) two C^*-algebraic quantum groups $\mathbb{G} := (\mathfrak{G}, \Delta_{\mathfrak{G}})$ and $\mathbb{H} := (\mathfrak{H}, \Delta_{\mathfrak{H}})$ (here, $\mathfrak{G}$ is a C^*-algebra and $\Delta_{\mathfrak{G}} \in \mathsf{Mor}(\mathfrak{G}, \mathfrak{G}\otimes_{\min}\mathfrak{G}) := \{\phi\colon \mathfrak{G} \to M(\mathfrak{G}\otimes_{\min}\mathfrak{G})\colon \phi \text{ non-degenerate } {}^*\text{-homomorphism}\}$ given in Theorem 2 of [73] (p. 44) and similarly for $(\mathfrak{H}, \Delta_{\mathfrak{H}})$);

(2) a bicharacter $u \in UM(\widehat{\mathfrak{G}} \otimes_{\min} \widehat{\mathfrak{H}})$ (unitary group of the multiplier algebra of $\widehat{\mathfrak{G}} \otimes_{\min} \widehat{\mathfrak{H}}$);

(3) two continuous right coactions $\delta^\alpha \in \mathsf{Mor}(\mathfrak{A}, \mathfrak{A} \otimes_{\min} \mathfrak{G})$ and $\delta^\beta \in \mathsf{Mor}(\mathfrak{B}, \mathfrak{B} \otimes_{\min} \mathfrak{H})$ of $\mathfrak{G}$ and $\mathfrak{H}$ on C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$, respectively.

The *spatial* twisted tensor product is then built out of two non-degenerate representations on the same Hilbert space $\mathcal{H}$

$$\begin{array}{ccc} \mathfrak{G} & \xrightarrow{\ \pi\ } & \mathcal{B}(\mathcal{H}) \\ & \overset{\rho}{\nearrow} & \\ \mathfrak{H} & & \end{array}$$

satisfying a certain commutation relation involving the bicharacter u and two unitary multipliers $W_{\mathfrak{G}} \in UM(\widehat{\mathfrak{G}} \otimes_{\min} \mathfrak{G})$ and $W_{\mathfrak{H}} \in UM(\widehat{\mathfrak{H}} \otimes_{\min} \mathfrak{H})$ giving rise to the quantum groups $\mathbb{G}$ and $\mathbb{H}$ according to Theorem 2 in [73] (p. 44). The commutation relation mimics the Heisenberg's CCR in the Weyl form, that is why (π, ρ) is called a *Heisenberg pair*. By using Baaj-Skandalis leg numbering notation (cf. [5]), the spatial twisted tensor product is defined as

$$\mathfrak{A} \boxtimes^u_{\min} \mathfrak{B} := \iota_{\mathfrak{A}}(\mathfrak{A}) \cdot \iota_{\mathfrak{B}}(\mathfrak{B}) \,,$$

where

$$\begin{array}{ccl} a & \longmapsto & [(\mathrm{id}_{\mathfrak{A}} \otimes \pi)(\delta^{\alpha}(a))]_{13} \\ \mathfrak{A} & \overset{\iota_{\mathfrak{A}}}{\hookrightarrow} & M(\mathfrak{A} \otimes_{\min} \mathfrak{B} \otimes_{\min} \mathcal{K}(\mathcal{H})) \\ \mathfrak{B} & \overset{\iota_{\mathfrak{B}}}{\hookrightarrow} & M(\mathfrak{A} \otimes_{\min} \mathfrak{B} \otimes_{\min} \mathcal{K}(\mathcal{H})) \\ b & \longmapsto & \left[(\mathrm{id}_{\mathfrak{B}} \otimes \rho)(\delta^{\beta}(b))\right]_{23} \end{array}$$

In [57], it is also shown that $\mathfrak{A} \boxtimes^u_{\min} \mathfrak{B}$ does not depend on the choice of the Heisenberg pair (π, ρ).

The point is that the construction significantly simplifies when the involved C^*-quantum groups are classical ones. In addition, it allows to discover and prove much more properties. Indeed, if $\mathbb{G} := (\mathcal{C}_0(G), \Delta_G)$ and $\mathbb{H} := (\mathcal{C}_0(H), \Delta_H)$, where G, H are locally compact abelian groups and

$$\begin{aligned} \Delta_G \colon \mathcal{C}_0(G) &\to \mathcal{C}_0(G^2) \\ f &\mapsto [(g, g') \mapsto f(gg')] \end{aligned}$$

is the usual co-multiplication of $\mathcal{C}_0(G)$ (analogously, for $\mathcal{C}_0(H)$), we have that

$$\begin{aligned} UM(\widehat{\mathcal{C}_0(G)} \otimes_{\min} \widehat{\mathcal{C}_0(H)}) =& UM(C^*(G) \otimes_{\min} C^*(H)) \\ =& UM(\mathcal{C}_0(\widehat{G}) \otimes_{\min} \mathcal{C}_0(\widehat{H})) = \mathcal{C}(\widehat{G} \times \widehat{H}, \mathbb{T}) \,, \end{aligned}$$

whence

$$u \in \mathsf{B}(\widehat{G} \times \widehat{H}) \subset \mathcal{C}(\widehat{G} \times \widehat{H}, \mathbb{T}) \,.$$

Furthermore, the coaction $\delta^{\alpha} \colon \mathfrak{A} \to M(\mathfrak{A} \otimes_{\min} \mathcal{C}_0(G)) \cong \mathcal{C}_b(G, \mathfrak{A})$ bijectively corresponds to a strongly continuous action α of G on $\mathfrak{A}$ by the formula $\alpha_g(a) := \delta^{\alpha}(a)(g) \in \mathfrak{A}$ $(g \in G)$, so that the triplet $(\mathfrak{A}, G, \alpha)$ is merely a C^*-system. We shall suppose $\mathfrak{A}$ unital from now on, so that δ^{α} is a unital *-homomorphism and $\mathbb{C}1_{\mathfrak{A}} \subset \mathfrak{A}^G$. In the same way, we get a (unital) C^*-system $(\mathfrak{B}, H, \beta)$. If G, H are also compact, $\mathfrak{A}$ and $\mathfrak{B}$ are respectively $\widehat{G}$-graded and $\widehat{H}$-graded, with $\widehat{G}, \widehat{H}$ necessarily discrete, and the above-mentioned Heisenberg-type commutation relation becomes

$$\rho_U(\sigma)\rho_V(\tau) = u(\sigma, \tau)\rho_V(\tau)\rho_U(\sigma), \qquad \sigma \in \widehat{G}, \tau \in \widehat{H}$$

where $\rho_U \colon \mathcal{C}(G) \to \mathcal{B}(\mathcal{H})$ (respectively, $\rho_V \colon \mathcal{C}(H) \to \mathcal{B}(\mathcal{H})$) is the *integrated form* of a suitable strongly continuous unitary representation of G (respectively, H). Notice that, by compactness of G, $\widehat{G} \subset \mathcal{C}(G)$ (even more, $\widehat{G}$ is a total subset of $\mathcal{C}(G)$, by Stone-Weierstrass density theorem).

The mapping $\mathbf{U}\colon \widehat{G} \ni \sigma \mapsto \rho_U(\sigma) = \int_G \sigma(g) U_g \mathrm{d}g \in \mathcal{U}(\mathcal{H})$ is a strongly continuous unitary representation of $\widehat{G}$ on $\mathcal{H}$, and similarly for $\mathbf{V}\colon \widehat{H} \ni \tau \mapsto \rho_V(\tau) \in \mathcal{U}(\mathcal{H})$, so that the Heisenberg-type relation boils down to

$$\mathbf{U}_\sigma \mathbf{V}_\tau = u(\sigma,\tau)\mathbf{V}_\tau \mathbf{U}_\sigma, \qquad \sigma \in \widehat{G}, \tau \in \widehat{H}. \tag{II.20}$$

In other words, an Heisenberg pair for $(\mathcal{C}_0(G), \mathcal{C}_0(H))$ is nothing else than a pair of unitary representations $(\mathbf{U},\mathbf{V})$ (of $\widehat{G}$ and $\widehat{H}$, respectively) acting on the same Hilbert space and satisfying relation (II.20). Lastly,

$$\begin{aligned}
\mathfrak{A} \boxtimes^u_{\min} \mathfrak{B} &= \{(\mathrm{id}_{\mathfrak{A}} \otimes \rho_U)(g \mapsto \alpha_g(a))_{13} (\mathrm{id}_{\mathfrak{B}} \otimes \rho_V)(h \mapsto \beta_h(b))_{23}\}_{a\in\mathfrak{A}, b\in\mathfrak{B}} \\
&= \left[\{(\mathrm{id}_{\mathfrak{A}} \otimes \rho_U)(a_\sigma \otimes \sigma)_{13} (\mathrm{id}_{\mathfrak{B}} \otimes \rho_V)(b_\tau \otimes \tau)_{23}\}_{\sigma\in\widehat{G}, \tau\in\widehat{H}}\right] \\
&= \left[\{(a_\sigma \otimes \mathbf{U}_\sigma)_{13}(b_\tau \otimes \mathbf{V}_\tau)_{23}\}_{\sigma\in\widehat{G}, \tau\in\widehat{H}}\right] \\
&= \left[\{a_\sigma \otimes b_\tau \otimes \mathbf{U}_\sigma \mathbf{V}_\tau\}_{\sigma\in\widehat{G}, \tau\in\widehat{H}}\right] \subset \mathfrak{A} \otimes_{\min} \mathfrak{B} \otimes_{\min} \mathcal{B}(\mathcal{H}).
\end{aligned}$$

To show that $\mathfrak{A} \boxtimes^u_{\min} \mathfrak{B}$ is isomorphic to $\mathfrak{A} \circledast_{\min} \mathfrak{B}$, we need Theorems 4.1, 4.2 and Equation (4.2) in [57] which will be condensed and reformulated in our classical group framework in the following statement. Henceforth, G and H will be compact abelian groups.

Theorem II.17.1
Let $(\pi, U, \mathcal{H})$ and $(\phi, V, \mathcal{L})$ be faithful covariant representations of $(\mathfrak{A}, G, \alpha)$ and $(\mathfrak{B}, H, \beta)$, respectively. Set $Z := (\rho_U \otimes \rho_V)(u)^* \in \mathcal{U}(\mathcal{H} \otimes \mathcal{L})$ (recall that $u \in UM(C^*(G) \otimes_{\min} C^*(H))$). If

$$\begin{cases} \Pi\colon a \hookrightarrow \pi(a) \otimes I_{\mathcal{L}}, \\ \Phi\colon b \hookrightarrow Z(I_{\mathcal{H}} \otimes \phi(b))Z^*, \end{cases}$$

then there exists a unique faithful representation $\Psi\colon \mathfrak{A} \boxtimes^u_{\min} \mathfrak{B} \hookrightarrow \mathcal{B}(\mathcal{H} \otimes \mathcal{L})$ s.t. each triangle of the diagram

$$\begin{array}{ccccc}
\mathfrak{A} & \xhookrightarrow{\iota_{\mathfrak{A}}} & \mathfrak{A} \boxtimes^u_{\min} \mathfrak{B} & \xhookleftarrow{\iota_{\mathfrak{B}}} & \mathfrak{B} \\
& {\scriptstyle\Pi}\searrow & \downarrow{\scriptstyle\Psi} & \swarrow{\scriptstyle\Phi} & \\
& & \mathcal{B}(\mathcal{H} \otimes \mathcal{L}) & &
\end{array}$$

commutes. In particular, $\Psi(a_\sigma \otimes b_\tau \otimes \rho_U(\sigma)\rho_V(\tau)) = (\pi_U \circledast \phi)(a_\sigma \otimes b_\tau)$ for every $\sigma \in \widehat{G}, \tau \in \widehat{H}$.

Proof.
The first part of the theorem corresponds to Theorems 4.1, 4.2 and Equation (4.2) in [57]. We are left to show the last equality. Firstly, notice that

$$u = \sum_{\widehat{G},\widehat{H}} u(\sigma,\tau) \mathbb{1}_\sigma \otimes \mathbb{1}_\tau \in \mathcal{C}(\widehat{G} \times \widehat{H}, \mathbb{T}) = UM(\mathcal{C}_0(\widehat{G} \times \widehat{H})).$$

In general, the finest common topology w.r.t. which the sum converges is the strict one on $\mathcal{C}_b(\widehat{G} \times \widehat{H}) = M(\mathcal{C}_0(\widehat{G} \times \widehat{H}))$: for every $f \in \mathcal{C}_0(\widehat{G} \times \widehat{H})$ and $\varepsilon > 0$, by the very definition of $\mathcal{C}_0(\widehat{G} \times \widehat{H})$ there must exist a finite set $F_\varepsilon \in \mathcal{F}(\widehat{G} \times \widehat{H})$ such that

$$\begin{aligned}
&\left\| \left(u - \sum_{F_\varepsilon} u(\sigma,\tau) \mathbb{1}_\sigma \otimes \mathbb{1}_\tau \right) f \right\|_{\infty, \widehat{G}\times\widehat{H}} = \\
&= \sup_{(x,y)\in\widehat{G}\times\widehat{H}} |(u(x,y) - u(x,y)\mathbb{1}_{F_\varepsilon}(x,y)) f(x,y)| = \|f\|_{\infty, F_\varepsilon} \leq \varepsilon.
\end{aligned}$$

As an element of $M(C^*(G) \otimes_{\min} C^*(H)) \subset (C^*(G) \otimes_{\min} C^*(H))'' \subset \mathcal{B}(L^2(G) \otimes L^2(H))$,

$$u = \sum_{\widehat{G},\widehat{H}} u(\sigma,\tau) \underbrace{\int_G \sigma(g)\lambda_g \mathrm{d}g}_{\rho_\lambda(\sigma)=\sigma\,*} \otimes \underbrace{\int_H \tau(h)\lambda_h \mathrm{d}h}_{\rho_\lambda(\tau)=\tau\,*}$$

where the sum is strictly convergent[8].

Now, $\rho_U \otimes \rho_V \colon C^*(G) \otimes_{\min} C^*(H) \to \mathcal{B}(\mathcal{H} \otimes \mathcal{L})$ is a non-degenerate representation (in other words, $\rho_U \otimes \rho_V \in \mathsf{Mor}(C^*(G) \otimes_{\min} C^*(H), \mathcal{K}(\mathcal{H} \otimes \mathcal{L})))$ and hence continuous if both $C^*(G) \otimes_{\min} C^*(H)$ and $\mathcal{B}(\mathcal{H} \otimes \mathcal{L})$ are endowed with the strict topology (on bounded subsets of $\mathcal{B}(\mathcal{H} \otimes \mathcal{L})$, it is nothing but the σ-strong* topology). Since $C^*(G) \otimes_{\min} C^*(H)$ is strictly dense in its multiplier algebra, $\rho_U \otimes \rho_V$ extends to a strict-strict continuous representation

$$\rho_U \otimes \rho_V \colon M(C^*(G) \otimes C^*(H)) \to \mathcal{B}(\mathcal{H} \otimes \mathcal{L})\,.$$

Also, notice that it is the restriction to $M(C^*(G) \otimes C^*(H))$ of the unique normal extension $(\rho_U \otimes \rho_V)'' \colon (C^*(G) \otimes C^*(H))'' \to \mathcal{B}(\mathcal{H} \otimes \mathcal{L})$. It follows that the sum

$$\mathcal{U}(\mathcal{H} \otimes \mathcal{L}) \ni Z^* = \sum_{\widehat{G},\widehat{H}} u(\sigma,\tau) \underbrace{\int_G \sigma(g) U_g \mathrm{d}g}_{\rho_U(\sigma)} \otimes \underbrace{\int_H \tau(h) V_h \mathrm{d}h}_{\rho_V(\tau)}$$

converges in the strict/σ-strong* topology of $\mathcal{B}(\mathcal{H} \otimes \mathcal{L})$. It is easy to see that $Z^*(\xi_\chi \otimes \eta_\zeta) = u(\chi,\zeta)\xi_\chi \otimes \eta_\zeta$ for every $\chi \in \widehat{G}, \zeta \in \widehat{H}$. At this stage, recall that $\iota_{\mathfrak{A}} \colon a_\sigma \mapsto a_\sigma \otimes \mathbb{1}_{\mathfrak{B}} \otimes \rho_U(\sigma)$ and $\iota_{\mathfrak{B}} \colon b_\tau \mapsto \mathbb{1}_{\mathfrak{A}} \otimes b_\tau \otimes \rho_V(\tau)$, therefore by the first part of the theorem,

$$\begin{cases} \Psi \colon a_\sigma \otimes \mathbb{1}_{\mathfrak{B}} \otimes \rho_U(\sigma) \mapsto [\pi(a_\sigma) \otimes I_{\mathcal{L}} : \xi_\chi \otimes \eta_\zeta \mapsto \pi(a_\sigma)\xi_\chi \otimes \eta_\zeta] \\ \Psi \colon \mathbb{1}_{\mathfrak{A}} \otimes b_\tau \otimes \rho_V(\tau) \mapsto [Z(I_{\mathcal{H}} \otimes \phi(b_\tau))Z^* \colon \xi_\chi \otimes \eta_\zeta \mapsto U(g_\tau)\xi_\chi \otimes \phi(b_\tau)\eta_\zeta] \end{cases}$$

Juxtaposing the two factors, we get

$$\Psi \colon a_\sigma \otimes b_\tau \otimes \rho_U(\sigma)\rho_V(\tau) \mapsto [\xi_\chi \otimes \eta_\zeta \mapsto \pi(a_\sigma)U(g_\tau)\xi_\chi \otimes \phi(b_\tau)\eta_\zeta]$$

Since the sets $\{\xi_\chi\}_{\chi \in \widehat{G}} \subset \mathcal{H}$ and $\{\eta_\zeta\}_{\zeta \in \widehat{H}} \subset \mathcal{L}$ are total, for every $\sigma \in \widehat{G}$ and $\tau \in \widehat{H}$ we have

$$\Psi(a_\sigma \otimes b_\tau \otimes \rho_U(\sigma)\rho_V(\tau)) = (\pi_U \mathbin{\textcircled{w}} \phi)(a_\sigma \mathbin{\textcircled{w}} b_\tau)$$

as bounded linear operators of $\mathcal{H} \otimes \mathcal{L}$. □

It is worth noticing that the only step of the proof where the covariance of $(\phi, V, \mathcal{L})$ takes place is when computing

$$\Phi(b_\tau)(\xi_\chi \otimes \eta_\zeta) = Z(I_{\mathcal{H}} \otimes \phi(b_\tau))Z^*(\xi_\chi \otimes \eta_\zeta) = U(g_\tau)\xi_\chi \otimes \phi(b_\tau)\eta_\zeta\,,$$

where the right-hand side perfectly makes sense even when ϕ is not covariant: g_τ is determined by the degree of b_τ only, while $\phi(b_\tau)$ can even be non-homogeneous. For this reason, in Theorem II.17.1 one can drop the hypothesis of covariance of $(\phi, V, \mathcal{L})$, bypass the definition of $Z \in \mathcal{U}(\mathcal{H} \otimes \mathcal{L})$ and directly *define* $\Phi \colon b_\tau \mapsto U(g_\tau) \otimes \phi(b_\tau)$, $\tau \in \widehat{H}$, which uniquely extends to a representation of $\mathfrak{B}$ as already observed. This chance makes it clear once more why our definition of $\pi_U \mathbin{\textcircled{w}} \phi$ does not actually need a unitary representation V of H on $\mathcal{L}$ s.t. (ϕ, V) is covariant. We are now ready for the first main result of this section.

[8]The symbol $*$ in the previous formula denotes the convolution product.

Theorem II.17.2
$\mathfrak{A} \boxtimes^u_{\min} \mathfrak{B} \cong \mathfrak{A} \textcircled{u}_{\min} \mathfrak{B}$.

Proof.
On the one hand, by Theorem II.17.1 and the above considerations, two faithful representations $(\pi, U) \in \mathsf{Cov}(\mathfrak{A}, G, \alpha)$ and $\phi \in \mathsf{Rep}(\mathfrak{B})$ give rise to a faithful representation Ψ of $\mathfrak{A} \boxtimes^u_{\min} \mathfrak{B}$. On the other hand, by our characterization of the spatial C^*-norm, the same (π, U) and ϕ give rise to a faithful representation $\pi_U \textcircled{u} \phi$ of $\mathfrak{A} \textcircled{u}_{\min} \mathfrak{B}$. Again by Theorem II.17.1,

$$\mathfrak{A} \boxtimes^u_{\min} \mathfrak{B} \cong \mathrm{im}(\Psi) = \mathrm{im}(\pi_U \textcircled{u} \phi) \cong \mathfrak{A} \textcircled{u}_{\min} \mathfrak{B}$$

and the proof is accomplished. □

The previous theorem also generalizes Theorem 6.1 in [57] (p. 31) which shows that when $\mathfrak{A}, \mathfrak{B}$ are $\mathbb{Z}_2$-graded C^*-algebras, $\mathfrak{A} \boxtimes^u_{\min} \mathfrak{B}$ is isomorphic to the Kasparov's skew-commutative tensor product (see [49]).
Let us turn to the *maximal* twisted tensor product defined in [70]. Again, a Heisenberg pair (π, ρ) is needed (actually, an anti-Heisenberg one) and the maximal twisted tensor product is defined as

$$\mathfrak{A} \boxtimes^u_{\max} \mathfrak{B} := j_{\mathfrak{A},\mathrm{u}}(\mathfrak{A}) \cdot j_{\mathfrak{B},\mathrm{u}}(\mathfrak{B}) \,,$$

where $(j_{\mathfrak{A},\mathrm{u}}, j_{\mathfrak{B},\mathrm{u}}, \mathcal{B}(\mathcal{H}_\mathrm{u}))$ is a *commutative representation* of $(\mathfrak{A}, \delta^\alpha)$ and $(\mathfrak{B}, \delta^\beta)$, that is

(1) $j_{\mathfrak{A},\mathrm{u}} \colon \mathfrak{A} \to \mathcal{B}(\mathcal{H}_\mathrm{u})$ and $j_{\mathfrak{B},\mathrm{u}} \colon \mathfrak{B} \to \mathcal{B}(\mathcal{H}_\mathrm{u})$ are non-degenerate representations,

(2) $[(j_{\mathfrak{A},\mathrm{u}} \otimes \pi)(\delta^\alpha(a)), (j_{\mathfrak{B},\mathrm{u}} \otimes \rho)(\delta^\beta(b))]_{\mathcal{B}(\mathcal{H}_\mathrm{u} \otimes \mathcal{H})} = 0 \ (a \in \mathfrak{A}, b \in \mathfrak{B})$

and it is an initial object in the category of all the commutative representations and their morphisms. Again, all technical difficulties in defining the maximal twisted tensor product vanish in the case when $\mathfrak{G} := (\mathcal{C}(G), \Delta_G)$ and $\mathfrak{H} := (\mathcal{C}(H), \Delta_H)$, where G, H are compact abelian groups and $\mathfrak{A}, \mathfrak{B}$ are unital. On the one hand, the anti-Heisenberg relation becomes

$$\mathbf{U}_\sigma \mathbf{V}_\tau = \overline{u(\sigma, \tau)} \mathbf{V}_\tau \mathbf{U}_\sigma, \qquad \sigma \in \widehat{G}, \tau \in \widehat{H} \,,$$

with $(\mathbf{U}, \mathbf{V})$ unitary representations of $\widehat{G}$ and $\widehat{H}$ respectively, acting on the same Hilbert space. On the other hand,

$$[j_{\mathfrak{A},\mathrm{u}}(a_\sigma) \otimes \mathbf{U}_\sigma, j_{\mathfrak{B},\mathrm{u}}(b_\tau) \otimes \mathbf{V}_\tau]_{\mathcal{B}(\mathcal{H}_\mathrm{u} \otimes \mathcal{H})} = 0, \qquad \sigma \in \widehat{G}, \tau \in \widehat{H} \,,$$

which reduces to

$$j_{\mathfrak{A},\mathrm{u}}(a_\sigma) j_{\mathfrak{B},\mathrm{u}}(b_\tau) = u(\sigma, \tau) j_{\mathfrak{B},\mathrm{u}}(b_\tau) j_{\mathfrak{A},\mathrm{u}}(a_\sigma), \qquad \sigma \in \widehat{G}, \tau \in \widehat{H} \,.$$

A possible choice for $j_{\mathfrak{A},\mathrm{u}}$ and $j_{\mathfrak{B},\mathrm{u}}$ are the bounded extensions of $\pi_\mathrm{u}(\cdot \textcircled{u} \mathbb{1}_\mathfrak{B})$ to $\mathfrak{A}$ and of $\pi_\mathrm{u}(\mathbb{1}_\mathfrak{A} \textcircled{u} \cdot)$ to $\mathfrak{B}$ respectively, where $\pi_\mathrm{u} := \bigoplus_{\mathcal{S}(\mathfrak{A}_o \textcircled{u} \mathfrak{B}_o)} \pi_f$ is the universal representation of $\mathfrak{A}_o \textcircled{u} \mathfrak{B}_o$. With this in mind, we obtain the second main result of this section.

Theorem II.17.3
$\mathfrak{A} \boxtimes^u_{\max} \mathfrak{B} \cong \mathfrak{A} \textcircled{u}_{\max} \mathfrak{B} \cong C^*(\mathfrak{A}_o \textcircled{u} \mathfrak{B}_o)$.

Proof.
$\mathfrak{A} \boxtimes^u_{\max} \mathfrak{B} = \pi_\mathrm{u}(\cdot \textcircled{u} \mathbb{1}_\mathfrak{B}) \cdot \pi_\mathrm{u}(\mathbb{1}_\mathfrak{A} \textcircled{u} \cdot) = \pi_\mathrm{u}(\mathfrak{A} \textcircled{u}_{\max} \mathfrak{B}) \cong \mathfrak{A} \textcircled{u}_{\max} \mathfrak{B}$, where the second equality is due to Theorem II.14.1. Moreover, by our characterization of the max-norm, $\mathfrak{A} \textcircled{u}_{\max} \mathfrak{B} \cong C^*(\mathfrak{A}_o \textcircled{u} \mathfrak{B}_o)$. □

All things considered, we reach the third (and last) main result of this section.

Theorem II.17.4
The involutive algebra $\mathfrak{A}_o \mathbin{\textcircled{u}} \mathfrak{B}_o$ admits a unique compatible C^*-norm if and only if either $\mathfrak{A}$ or $\mathfrak{B}$ is nuclear or, equivalently, if and only if $\mathfrak{A}^G$ or $\mathfrak{B}^H$, in which case $\mathfrak{A} \mathbin{\textcircled{u}}_{\min} \mathfrak{B} \cong \mathfrak{A} \mathbin{\textcircled{u}}_{\max} \mathfrak{B}$.

Proof.
The "if" part is a straightforward application of Theorem II.17.2 and Theorem II.17.3, together with point (2) of Corollary 7.7 in [70] (p. 314). The "only if" part is promptly explained. By $\mathfrak{A} \mathbin{\textcircled{u}}_{\min} \mathfrak{B} \cong \mathfrak{A} \mathbin{\textcircled{u}}_{\max} \mathfrak{B}$, we have that $\mathfrak{A}^G \otimes_{\min} \mathfrak{B}^H \cong \mathfrak{A}^G \otimes_{\max} \mathfrak{B}^H$. Then, by Theorem 3.8.7 in [88] (p. 104) either $\mathfrak{A}^G$ or $\mathfrak{B}^H$ is nuclear, or equivalently (by Theorem 4.5.2 in [88], p. 134) either $\mathfrak{A}$ or $\mathfrak{B}$ is nuclear. In such a case, by Theorem II.12.3 and Proposition II.14.2, $\mathfrak{A}_o \mathbin{\textcircled{u}} \mathfrak{B}_o$ admits a unique compatible C^*-norm. □

Remark II.17.5
The theorem above significantly generalizes Proposition II.12.2. As a last consideration, we observe that both $\mathfrak{A} \mathbin{\textcircled{u}}_{\min} \mathfrak{B}$ and $\mathfrak{A} \mathbin{\textcircled{u}}_{\max} \mathfrak{B}$ can be regarded as Rieffel deformations (in Kasprzak's sense, [50]) of their respective usual tensor products w.r.t. a suitable 2-cocycle, see Theorem 6.2 in [57] (p. 33) for the spatial case, Theorem 7.10 in [70] (p. 316) for the maximal one.

II.18. The Klein transformation

The Klein-Jordan-Wigner transformation (here simply called Klein transformation) plays a crucial role in quantum theories: it allows to pass from operators acting on a common Hilbert space and enjoying the Canonical Anticommutation Relations (i.e. Fermi elementary fields) to others, acting on the same space but now satisfying the Canonical Commutation Relations (i.e. Bose elementary fields), see *e.g.* [103]. The implementation of such a transformation is that introduced in (II.13) and (II.14) to build the twisted tensor product of representations. When implementable, it realizes a *-isomorphism between $\mathfrak{A} \mathbin{\textcircled{u}}_{\min} \mathfrak{B}$ and $\mathfrak{A} \otimes_{\min} \mathfrak{B}$ (see *e.g.* [49], [31], [57]), then "straightening" the twisted product. Not only that, it may have natural applications in quantum probability and information theory, since it preserves the product states as we will show below. Hence, it turns out logical to investigate the chance of implementing such a transformation in a general twisted setting.
We start with a C^*-system $(\mathfrak{A}, G, \alpha)$, and say that the action α_g is *inner* if the following properties are satisfied:

(a) there is a unitary representation $G \ni g \mapsto u(g) \in \mathcal{U}(\mathfrak{A})$ such that

$$\alpha_g(a) = u(g)au(g^{-1}), \quad a \in \mathfrak{A},\ g \in G\ :$$

(b) such a representation $G \ni g \mapsto u(g) \in \mathcal{U}(\mathfrak{A})$ is continuous when $\mathfrak{A}$ is equipped with the seminorms

$$p_{\pi,\xi}(a) := \|\pi(a)\xi\|, \quad \pi \in \mathsf{Rep}(\mathfrak{A}),\ \xi \in \mathcal{H}_\pi\,.$$

Let $\omega \in \mathcal{S}_G(\mathfrak{A})$, and consider its GNS representation $(\pi_\omega, \mathcal{H}_\omega, \xi_\omega)$. By assumption, the action α is implemented by the representation $g \mapsto \pi_\omega(u(g))$, which is continuous in the strong operator topology, hence by Proposition II.15.3 we get

$$\|x\|_{\min} = \sup_{\substack{\omega \in \mathcal{S}_G(\mathfrak{A}) \\ \varphi \in \mathcal{S}_H(\mathfrak{B})}} \left\| \left((\pi_\omega)_{\pi_\omega(u)} \mathbin{\textcircled{u}} \pi_\varphi \right)(x) \right\|, \quad x \in \mathfrak{A} \mathbin{\textcircled{u}}_{\min} \mathfrak{B}$$

for every other C^*-system $(\mathfrak{B}, H, \beta)$. We also note that $u(G) \subset \mathfrak{A}^G$ because

$$u(h) = u(g)u(h)u(g^{-1}) = \alpha_g(u(h))\,, \quad g, h \in G\,.$$

After setting $u_\tau := u(g_\tau)$, $\tau \in \widehat{H}$ and g_τ defined in (II.13), we have

$$u_\iota = u(e_G) = \mathbb{1}_{\mathfrak{A}},\ (u_\tau)^* = u_{\tau^{-1}},\ u_{\tau_1}u_{\tau_2} = u_{\tau_1\tau_2}\,, \quad \tau, \tau_1, \tau_2 \in \widehat{H}\,,$$

that is $\{u_\tau \colon \tau \in \widehat{H}\} \subset \mathcal{U}(\mathfrak{A}^G)$ realizes a representation of $\widehat{H}$ in $\mathcal{U}(\mathfrak{A}^G)$.
On the generators in $\mathfrak{A}_o \odot \mathfrak{B}_\tau$, define

$$\mathfrak{A}_o \odot \mathfrak{B}_\tau \ni a \textcircled{u} b \mapsto \kappa_o(a \textcircled{u} b) := (au_\tau) \otimes b \in \mathfrak{A}_o \otimes \mathfrak{B}_o\,, \tag{II.21}$$

and extend it by linearity to the whole $\mathfrak{A}_o \textcircled{u} \mathfrak{B}_o$. As seen in the proof of Proposition II.8.1, κ_o is well defined.

Theorem II.18.1
Let $(\mathfrak{A}, G, \alpha)$ and $(\mathfrak{B}, H, \beta)$ be C^*-systems such that α is inner. Then, the map in (II.21) extends to a *-isomorphism

$$\kappa \colon \mathfrak{A} \textcircled{u}_{\min} \mathfrak{B} \to \mathfrak{A} \otimes_{\min} \mathfrak{B}$$

satisfying

(i$_{\rm L}$) $\kappa^{\rm t}\big(\psi_{\omega,\varphi}\big) = \omega \times \varphi$ for every $\omega \in \mathcal{S}(\mathfrak{A})$, $\omega \in \mathcal{S}_H(\mathfrak{B})$;

(ii) $(\alpha \otimes_{\min} \beta) \circ \kappa = \kappa \circ \big(\alpha \textcircled{u}_{\min} \beta\big)$.

Proof.
We sketch the proof, leaving the algebraic details to the reader.
For $x, y \in \mathfrak{A}_o \textcircled{u} \mathfrak{B}_o$, we first notice that

$$\kappa_o(xy) = \kappa_o(x)\kappa_o(y)\,, \quad \kappa_o(x^*) = \kappa_o(x)^\dagger\,.$$

If $\omega \in \mathcal{S}(\mathfrak{A})$, $\varphi \in \mathcal{S}_H(\mathfrak{B})$, $a \in \mathfrak{A}_o$ and $b \in \mathfrak{B}_o$ is homogeneous, we compute:

$$\psi_{\omega,\varphi}(\kappa_o(a \textcircled{u} b)) = \psi_{\omega,\varphi}\big((au_{\partial b}) \otimes b\big) = \omega(a)\varphi(b)\delta_{\partial b,\iota} = (\omega \times \varphi)(a \textcircled{u} b) \tag{II.22}$$

by recalling that $u_\iota = \mathbb{1}_{\mathfrak{A}}$. Summarising, κ_o is a *-isomorphism between $\mathfrak{A}_o \textcircled{u} \mathfrak{B}_o$ and $\mathfrak{A}_o \otimes \mathfrak{B}_o$ sending product states with invariant right marginal onto product states of the same form. By the above considerations and applying (II.22), we also get

$$\begin{aligned}\|\kappa_o(x)\|_{\min,\otimes} &= \sup_{\substack{\omega\in\mathcal{S}_G(\mathfrak{A})\\ \varphi\in\mathcal{S}_H(\mathfrak{B})}} \big\|\pi_{\psi_{\omega,\varphi}}(\kappa_o(x))\big\| \\ &= \sup_{\substack{\omega\in\mathcal{S}_G(\mathfrak{A})\\ \varphi\in\mathcal{S}_H(\mathfrak{B})}} \big\|\big((\pi_\omega)_{\pi_\omega(u)} \textcircled{u} \pi_\varphi\big)(x)\big\| = \|x\|_{\min,\textcircled{u}}\end{aligned}$$

for each $x \in \mathfrak{A}_o \textcircled{u} \mathfrak{B}_o$. Therefore, κ_o realizes an isometric *-isomorphism (w.r.t the corresponding min-norms) between $\mathfrak{A}_o \textcircled{u} \mathfrak{B}_o$ and $\mathfrak{A}_o \otimes \mathfrak{B}_o$, the extension of which realizes a *-isomorphism between $\mathfrak{A} \textcircled{u}_{\min} \mathfrak{B}$ and $\mathfrak{A} \otimes_{\min} \mathfrak{B}$.
We now notice that (i$_{\rm L}$) directly follows from (II.22). As regards (ii), the product action $\alpha \times \beta$, is meaningful on both the tensor products (see Corollary II.15.4, Proposition II.11.2). Therefore, it is enough to check (ii) on elements of the form $a \textcircled{u} b$, b homogeneous. For such a purpose, we

recall that the u_τ's are α-invariant and that $\partial(\beta_h(b)) = \partial b$ for b homogeneous and $h \in H$, so we compute:

$$\begin{aligned}(\alpha_g \otimes_{\min} \beta_h)\big(\kappa(a \circledast b)\big) =& (\alpha_g \otimes_{\min} \beta_h)\big((au_{\partial b}) \otimes b\big) = \alpha_g(au_{\partial b}) \otimes \beta_h(b)\\ =& \alpha_g(a)u_{\partial b} \otimes \beta_h(b) = \alpha_g(a)u_{\partial\beta_h(b)} \circledast \beta_h(b)\\ =& \kappa\big((\alpha_g \circledast_{\min} \beta_h)(a \circledast b)\big), \quad g \in G, h \in H\end{aligned}$$

and the proof is accomplished. $\square$

The case corresponding to an inner action β on $\mathfrak{B}$ is handled exactly in the same way. Indeed, if β is implemented by a unitary representation $H \ni h \mapsto v(h) \in \mathcal{U}(\mathfrak{B})$, by using the construction in (II.14) and setting $v_\sigma := v({}_\sigma h)$ $(\sigma \in \widehat{G})$, now the Klein transformation assumes the form

$$\mathfrak{A}_\sigma \odot \mathfrak{B}_o \ni a \circledast b \mapsto \kappa_o(a \circledast b) := a \otimes (v_\sigma b) \in \mathfrak{A}_o \otimes \mathfrak{B}_o\,.$$

for each $\sigma \in \widehat{G}$. The new Klein transformation again satisfies (ii) in Theorem II.18.1, and

(i$_\mathrm{R}$) $\kappa^{\mathrm{t}}\big(\psi_{\omega,\varphi}\big) = \omega \times \varphi$ for every $\omega \in \mathcal{S}_G(\mathfrak{A})$, $\omega \in \mathcal{S}(\mathfrak{B})$

instead of (i$_\mathrm{L}$).

Chapter III

Symmetric states for Klein C^*-chains

III.1. Introduction

In Classical Probability, a sequence of random variables $(X_j)_{j\in\mathbb{N}}$ is said to be *exchangeable* (or *symmetric*) when the joint distribution of the sequence is invariant under all the permutations which swap the indices of a *finite* number of variables X_j's. Explicitly, if for every $j \in \mathbb{N}$

$$X_j \colon (\Omega, \mathcal{F}, \mathbb{P}) \to (E, \mathcal{E})$$

is a r.v. from a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ consisting of a set Ω (*sample space*), a σ-algebra $\mathcal{F}$ on Ω (*event space*) and a positive, normalized, countably additive measure $\mathbb{P}$ on $\mathcal{F}$ (*probability measure*), to a measure space $(E, \mathcal{E})$ where $\mathcal{E}$ is a σ-algebra on the set E (*state/value space*), the sequence $(X_j)_{j\in\mathbb{N}}$ is exchangeable if, for every fixed $n \in \mathbb{N}$,

$$\mathbb{P}((X_{\sigma(1)}, \dots, X_{\sigma(n)}) \in A) = \mathbb{P}((X_1, \dots, X_n) \in A)$$

for any permutation $\sigma \in \mathfrak{S}_n$ and any set A belonging to the product σ-algebra $\mathcal{E}^n$. As customary, if $\mathbf{X} := (X_1, \dots, X_n)$ is the vector r.v. associated to the first n terms of the sequence, "$\mathbb{P}((X_1, \dots, X_n) \in A)$„ stays for $\mathbb{P}(\mathbf{X}^{-1}(A)) = \mu_{\mathbf{X}}(A)$ where $\mu_{\mathbf{X}} \colon \mathcal{E}^n \to [0,1]$ is the *joint probability distribution* of $\mathbf{X}$, namely the pushforward measure of $\mathbb{P}$ through the measurable map $\mathbf{X}$. In symbols, for each $n \in \mathbb{N}$, $\mathfrak{S}_n \curvearrowright \mathcal{M}_1(E^n)$ and $\mu_{\mathbf{X}} \in \mathcal{M}_1(E^n)^{\mathfrak{S}_n}$, the fixed point family of the *symmetric* probability measures in $\mathcal{M}_1(E^n)$. The simplest example of exchangeable sequences are the ones consisting of independent and identically distributed (i.i.d., for short) random variables: in such a case, $\mu_{\mathbf{X}} = \bigotimes_{j=1}^{n} \nu$ for every $\mathbf{X} = (X_1, \dots, X_n)$, where $\nu := \mu_{X_j} \in \mathcal{M}_1(E)$ for any $j \in \mathbb{N}$. De Finetti's pioneering work [21] in 1931 shows that if $(X_j)_{j\in\mathbb{N}}$ is *any* exchangeable sequence of Bernoulli random variables, then there always exists $\mu \in \mathcal{M}_1([0,1])$ satisfying

$$\mathbb{P}(X_1 = \varepsilon_1, \dots, X_n = \varepsilon_n) = \int_0^1 \prod_{j=1}^{n} \beta_p(\varepsilon_j)\, \mathrm{d}\mu(p), \quad (\varepsilon_j)_{j=1}^n \in \{0,1\}^n, n \in \mathbb{N}$$

where $\beta_p \in \mathcal{M}_1(\{0,1\})$ is the Bernoulli probability mass function with expected value $p \in [0,1]$ (see [51] for a recent, simple and self-contained proof of De Finetti's result). Loosely speaking, the distribution of a $\{0,1\}$-valued exchangeable sequence is merely a "mixture" (with respect to some, not necessarily unique, measure $\mu \in \mathcal{M}_1([0,1])$) of sequences of i.i.d. Bernoulli random variables. Here, "μ-mixture" informally indicates the compound of two distinct distributions: the joint distribution of n independent Bernoulli r.v.'s with fixed expectation p and the distribution

μ of the parameter p itself.

One of the most general versions of De Finetti theorem in classical probability was undoubtedly obtained by Hewitt and Savage in 1953 (see [46]). To illustrate it, let us start from slightly afar. If E is a compact, Hausdorff topological space and $n \in \mathbb{N}$ is fixed, the Riesz-Markov-Kakutani representation theorem yields an isometric embedding of the (Borel, regular, positive) probability measures $\mathcal{M}_1(E^n)$ into $\mathcal{C}(E^n)^*$, the topological dual of the continuous $\mathbb{C}$-valued functions on E^n. On the other hand, the unit ball $B_{\mathcal{C}(E^n)^*}$ of $\mathcal{C}(E^n)^*$ is compact w.r.t. the weak-* topology (the coarsest one making evaluation functionals on $\mathcal{C}(E^n)^*$ continuous), by the Banach-Alaoglu-Bourbaki theorem and the convex set $\mathcal{M}_1(E^n)^{\mathfrak{S}_n}$ of all symmetric probability measures on E^n is weakly-* closed in $B_{\mathcal{C}(E^n)^*}$, hence compact. We might then consider the space M of all (Borel, regular, positive) probability measures on $\mathcal{M}_1(E^n)^{\mathfrak{S}_n}$ and deduce that if $m \in M$ is supported by the family of product measures $\mathcal{P}_n := \left\{ \bigotimes_{j=1}^{n} \nu \colon \nu \in \mathcal{M}_1(E) \right\}$ its *barycenter* $b_m := \int\limits_{\mathcal{M}_1(E^n)^{\mathfrak{S}_n}} \mu \, \mathrm{d}m(\mu) = \int\limits_{\mathcal{P}_n} \mu \, \mathrm{d}m(\mu)$ is symmetric, i.e. it belongs to $\mathcal{M}_1(E^n)^{\mathfrak{S}_n}$. Here, the definition of b_m stands for

$$f(b_m) = \int\limits_{\mathcal{P}_n} f(\mu)\mathrm{d}m(\mu) \text{ for every } f \colon \mathcal{M}_1(E^n)^{\mathfrak{S}_n} \to \mathbb{C} \text{ continuous and affine}.$$

The barycenter b_m is therefore nothing more than an average of product measures, weighted by $m \in M$, i.e. a generalized convex combination. Coming back to random variables, the Kolmogorov theorem guarantees that given any $\nu \in \mathcal{M}_1(E)$, there necessarily exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a sequence $(Y_i)_{i \in \mathbb{N}}$ of r.v.'s on it (read "discrete-time stochastic process"), with common state space $(E, \mathcal{E})$ s.t. $\bigotimes_{j=1}^{n} \nu = \mu_{(Y_1, \dots, Y_n)}$ for each $n \in \mathbb{N}$. In particular, it results that the Y_i's are i.i.d., with probability distribution $\nu \in \mathcal{M}_1(E)$, and then the above discussion translates into the fact that weighted averages of joint distributions of n i.i.d. random variables are always symmetric measures. To summarize the picture, if $E^{\mathbb{N}} := \prod_{j \in \mathbb{N}} E$ is the product of a countable number of copies of E (hence, compact by the Tychonoff theorem, and Hausdorff) and $\mathfrak{S}$ is the *finitary* symmetric group on $\mathbb{N}$, consisting of the permutations of the natural numbers $\mathbb{N}$ leaving fixed all but a finite number of elements, then $\mathfrak{S}$ acts on $E^{\mathbb{N}}$ in a natural way and any discrete-time stochastic process $(X_j)_{j \in \mathbb{N}}$ having finite-dimensional distributions of the form $\mu_{(X_1, \dots, X_n)} := \int\limits_{\mathcal{M}_1(E^{\mathbb{N}})^{\mathfrak{S}}} \omega \, \mathrm{d}m(\omega)$, with m probability measure on $\mathcal{M}_1(E^{\mathbb{N}})^{\mathfrak{S}}$ supported (or even, pseudo-supported) by $\mathcal{P} := \left\{ \bigotimes_{j \in \mathbb{N}} \nu \colon \nu \in \mathcal{M}_1(E^{\mathbb{N}}) \right\} \subset \mathcal{M}_1(E^{\mathbb{N}})^{\mathfrak{S}}$ is exchangeable. One might naturally ask: are there other exchangeable processes? Hewitt-Savage theorem gives a negative answer. Precisely, it asserts that $\mathcal{M}_1(E^{\mathbb{N}})^{\mathfrak{S}}$ is a *simplex* (in the sense of Choquet, see [98]) in $(\mathcal{C}(E^{\mathbb{N}})^*, \tau_{w^*})$ whose extremal boundary is closed and coincides exactly with the family of product states $\mathcal{P}$. Moreover, every $\mu \in \mathcal{M}_1(E^{\mathbb{N}})^{\mathfrak{S}}$ is the barycenter of a unique probability measure m on $\mathcal{M}_1(E^{\mathbb{N}})^{\mathfrak{S}}$ pseudo-supported by $\mathcal{P}$ and maximal w.t.r. to a suitable partial ordering $\prec$.

A first non-commutative extension of Hewitt-Savage result for infinite minimal C^*-tensor products $\mathfrak{A}$ of a fixed unital C^*-algebra $\mathfrak{B}$ was given by E. Størmer in 1969 in a noteworthy paper, [76]. Here, it is shown that the symmetric states of $\mathfrak{A}$ form a Choquet simplex $\mathcal{S}_{\mathfrak{S}}(\mathfrak{A})$

with closed extremal boundary (namely, a Bauer simplex) consisting exactly of the products of infinitely many copies of a single state on $\mathfrak{B}$, previously constructed by Guichardet in [43]. In hindsight, if $\mathfrak{B}$ is abelian, Størmer's theorem reduces to the Hewitt–Savage version. Around the same time, a maximal counterpart of Størmer's result have been published by Hulanicki and Phelps in [47] (we thank Lorenzin A. for notifying us about this work). We refer the reader to [18] (and the references cited therein) for versions of De Finetti's result in several other settings, including some in Free Probability and Quantum Information Theory. Here, we limit ourselves to illustrate the path started in [18] and partially traced in [31] with the aim of generalizing Størmer's theorem to *twisted* minimal C^*-tensor products, defined and thoroughly analyzed in Chapter II.

By exploiting the Jordan–Klein–Wigner transformation (see [105], Ex. XIV), in [18] Crismale and Fidaleo deeply study the symmetric states of the Canonical Anticommutation Relations C^*-algebra $\mathrm{CAR}(J)$, generated by Fermi *annihilators* and *creators* labelled by an arbitrary set J. One of the main tools in Størmer's theorem is the asymptotic abelianness property w.r.t. the finitary symmetric group $\mathfrak{S}$. In the CAR algebra, this property is not satisfied, due to the anticommutation relations between spatially separated operators. As a consequence, the results relative to the structure of symmetric states in [76] cannot be directly imported in [18] and a new approach turns out to be necessary. The crucial point in their work is the proof that each symmetric state on $\mathrm{CAR}(J)$ must be *even*, i.e. invariant under the parity involutive *-automorphism naturally acting on the algebra. This property is exploited throughout the paper in order to obtain a De Finetti-like ergodic decomposition for symmetric states. In particular, they characterize the ergodic (i.e. extremal symmetric) states and show that every symmetric state is the barycenter of a unique maximal probability measure which is pseudo-supported on the ergodic states. In addition, they prove that the extremal states form a weakly-* closed subset and determine the type of von Neumann factors generated by the extremal states. In [31], the CAR algebra model is incorporated in a much more general setting, the one of infinite, $\mathbb{Z}_2$-twisted, minimal C^*-tensor products (there called *Fermi* C^*-tensor products). Once fixed a $\mathbb{Z}_2$-graded C^*-algebra $\mathfrak{B}$, the infinite (minimal) Fermi C^*-tensor product $\mathfrak{A} := \textcircled{\scriptsize F}_{n\in\mathbb{N}} \mathfrak{B}$ is built via a direct limit procedure over tensor products of finitely many copies of $\mathfrak{B}$, twisted by the Fermi bicharacter $u_{\mathtt{F}}(x,y) = (-1)^{xy}$, $x, y \in \mathbb{Z}_2 = \{0,1\}$ as explained in the previous chapter. In [31], Fidaleo shows that

- $\mathcal{S}_{\mathfrak{S}}(\mathfrak{A})$ is a Bauer simplex whose boundary $\mathcal{E}_{\mathfrak{S}}(\mathfrak{A}) := \mathrm{Ext}(\mathcal{S}_{\mathfrak{S}}(\mathfrak{A}))$ consists exactly of the product states of the form $\prod_{n\in\mathbb{N}} \psi$, with $\psi \in \mathcal{S}_{\mathbb{Z}_2}(\mathfrak{B}) \cong \mathcal{S}(\mathfrak{B}^{\mathbb{Z}_2})$;
- each $\varphi \in \mathcal{S}_{\mathfrak{S}}(\mathfrak{A})$ admits a unique maximal probability measure μ_φ, pseudo-supported on $\mathcal{E}_{\mathfrak{S}}(\mathfrak{A})$, for which $\varphi = \int_{\mathcal{S}_{\mathfrak{S}}(\mathfrak{A})} \omega \, \mathrm{d}\mu_\varphi(\omega)$;
- if $\mathfrak{B}$ is also separable, then μ_φ is supported on $\mathcal{E}_{\mathfrak{S}}(\mathfrak{A})$ and $\varphi = \int_{\mathcal{E}_{\mathfrak{S}}(\mathfrak{A})} \omega \, \mathrm{d}\mu_\varphi(\omega)$.

Now that a solid theory of twisted tensor products is available from Chapter II, we might ask ourselves at which extent De Finetti theorem is feasible on a general infinite twisted C^*-tensor product $\mathfrak{A}$ arising from a single C^*-system $(G, \mathfrak{B}, \beta)$, where a compact abelian group G acts on $\mathfrak{B}$ via β (we shall call it *twisted C^*-chain*). The main goal of the present chapter is to show that, incidentally, the results achieved by Størmer and Fidaleo are two of the only *three* possible cases in which a full-fledged ergodic decomposition of symmetric states can be performed in general. The third arising case, never addressed before, is the (minimal) *Klein C^*-chain* $\mathfrak{A} := \textcircled{\scriptsize K}_{n\in\mathbb{N}} \mathfrak{B}$

associated to a C^*-system $(\mathfrak{B}, K_4, \beta)$, where $K_4 := \mathbb{Z}_2 \times \mathbb{Z}_2$ is the Klein 4-group acting on a *nuclear* C^*-algebra $\mathfrak{B}$ and the twist is produced by the bicharacter $u_{\mathtt{K}}(\mathbf{x}, \mathbf{y}) := (-1)^{\mathbf{x}^t \mathbf{y}}$, $\mathbf{x}, \mathbf{y} \in K_4$. No other De Finetti-like theorems can be deduced for group-twisted models in full generality (i.e. without imposing other restrictions). It is worth explaining *where* this extraordinary forcing comes from. We correct an inaccuracy in [31] (and consequently in [18]), where it is stated that $\mathfrak{S}$ acts on a Fermi C^*-chain $\underset{n\in\mathbb{N}}{\textcircled{\scriptsize F}}\mathfrak{B}$ via *-automorphisms given on the elementary tensor products by simple permutations of the indices. This is actually incorrect: we show that a transposition (namely, a 2-cycle in $\mathfrak{S}$) acts as an involutive *-automorphism on a minimal twisted C^*-chain if and only if the twisting bicharacter u is *skew-symmetric* and the flip map Φ which simply swaps two indices is substituted by the "corrected" flip Φ_u described in Proposition II.7.5 of the previous chapter. On the contrary, Φ can be at most a *-anti-automorphism, in case $\mathfrak{B}$ is abelian and u is symmetric. This fact leads to the analysis of non-degenerate, skew-symmetric bicharacter on discrete abelian groups, to which the trivial bicharacter on the trivial group (0), the Fermi bicharacter $u_{\mathtt{F}}$ on $\mathbb{Z}_2$ and the Klein bicharacter $u_{\mathtt{K}}$ on K_4 evidently belong (all of them are symmetric as well). Using techniques in ergodic theory exploited in [18] and [31], we are then able to show that a symmetric state must be invariant under a subgroup of the original acting group, precisely the annihilator $\Delta_+^\perp$ of the isotropy group $\Delta_+ := \{\sigma \colon u(\sigma, \sigma) = 1\}$ (see Section II.6). Furthermore, if the restriction of u to $\Delta_+ \times \Delta_+$ is identically 1, every symmetric state is automatically $\mathfrak{S}$-abelian (see definition below), a condition which is necessary and sufficient to endow the weakly-* compact, convex family of symmetric states $\mathcal{S}_{\mathfrak{S}}(\mathfrak{A})$ with the structure of a Choquet simplex. The requirement of triviality of the non-degenerate, skew-symmetric bicharacter u on $\Delta_+ \times \Delta_+$ forces it to be one of the three mentioned above: 1, $u_{\mathtt{F}}$ and $u_{\mathtt{K}}$. In light of this result, all the proofs in [31] can be effortlessly corrected by substituting the untwisted flip Φ with the twisted version Φ_u. Moreover, a new model can be now examined in depth: the Klein C^*-chains. This case requires an even more thorough look. For $u = u_{\mathtt{K}}$, $\Delta_+^\perp = \langle(1,1)\rangle = \{(0,0), (1,1)\}$ (the diagonal of K_4, an index-2 subgroup), hence every symmetric state must be $\langle(1,1)\rangle$-invariant. It turns out that if $(\mathfrak{B}, K_4, \beta)$ is a C^*-system and $\omega, \varphi \in \mathcal{S}_{\langle(1,1)\rangle}(\mathfrak{B})$, then $\omega \times \varphi$ is a state on the twisted tensor product $\mathfrak{B} \circledast \mathfrak{B}$ and $\|\cdot\|_{\langle(1,1)\rangle} := \sup\limits_{\omega,\varphi\in\mathcal{S}_{\langle(1,1)\rangle}(\mathfrak{B})} \|\pi_{\omega\times\varphi}(\cdot)\|$ defines a $(\beta \times \beta)$-compatible C^*-norm on $\mathfrak{B} \circledast \mathfrak{B}$, in general intermediate between the min and max-norm. Since $\mathfrak{S}$ acts as *-automorphisms on the *minimal* Klein C^*-chain which coincides with the chain obtained from the $\|\cdot\|_{\langle(1,1)\rangle}$-completion whenever $\mathfrak{B}$ is nuclear (in view of Theorem II.17.4), we restrict the investigation to this case and achieve a new De Finetti-like result:

Let $(\mathfrak{A}, \mathfrak{S})$ be the C^*-system associated to a unital, K_4-graded, nuclear C^*-algebra $\mathfrak{B}$. Then, for each $\varphi \in \mathcal{S}_{\mathfrak{S}}(\mathfrak{A})$, there exists a unique maximal $\mu_\varphi \in \mathcal{M}_1(\mathcal{S}_{\mathfrak{S}}(\mathfrak{A}))$ s.t.

$$\varphi(a) = \int\limits_{\mathcal{S}_{\mathfrak{S}}(\mathfrak{A})} \omega(a)\,\mathrm{d}\mu_\varphi(\omega), \qquad a \in \mathfrak{A}\,. \tag{III.1}$$

In particular, μ_φ is pseudo-supported by $\mathcal{E}_{\mathfrak{S}}(\mathfrak{A}) = \left\{\prod\limits_{n\in\mathbb{N}} \psi\right\}_{\psi\in\mathcal{S}_{\Delta_+}(\mathfrak{B})}$ i.e. $\mu_\varphi(B) = 1$ for every $B \in \mathcal{B}_0(\mathcal{S}_{\mathfrak{S}}(\mathfrak{A}))$ containing $\mathcal{E}_{\mathfrak{S}}(\mathfrak{A})$. The relative weak-* topology on the unit ball $B_{\mathfrak{A}^*}$ of $\mathfrak{A}^*$ is metrizable if and only if $\mathfrak{B}$ is separable, in which case μ_φ is supported by $\mathcal{E}_{\mathfrak{S}}(\mathfrak{A})$ and Equation III.1 becomes $\varphi(a) = \int\limits_{\mathcal{E}_{\mathfrak{S}}(\mathfrak{A})} \omega(a)\,\mathrm{d}\mu_\varphi(\omega), \qquad a \in \mathfrak{A}.$

After a necessary background of ergodic theory of C^*-systems in Section III.2, we describe the construction of the infinite, minimal, twisted C^*-tensor product, which we shall call *twisted*

(C^*-)chain, in Section III.3. The flip map Φ_u is the base for the study of the action of $\mathfrak{S}$ on a chain: its properties are developed and compared to the ones of Φ in Section III.4. Due to the suitability of skew-symmetric bicharacters for our analysis, we devote an entire section to the classification of non-degenerate, skew-symmetric bicharacters on finite abelian groups: this is the main result in Section III.5. It has turned out to be really hard to find in literature, but at last we have been able to attribute it to Zolotykh, see [82] (the article is in Russian, and an English rendition of its main results can be read in Subsection 3.2 of [2], pp. 4230-4232). In view of Section III.4, we are then ready to establish a well-defined action of $\mathfrak{S}$ on a twisted chain in Section III.6 and describe its ergodic properties in Section III.7. In particular, $\mathfrak{S}$-abelianness property of $\mathcal{S}_{\mathfrak{S}}(\mathfrak{A})$ is guaranteed in general if and only if one of three cases occur (see Corollary III.7.2 and Proposition III.7.4), as mentioned above. We summarize the results obtained by Størmer in [76] in Subsection III.7.1 and the ones achieved by Fidaleo in [31] in Subsection III.7.2. The third possible model, the Klein chain, is thoroughly investigated in Section III.8. We conclude the present chapter with three applications to this new model: K_4 acting faithfully on the continuous functions $\mathcal{C}(\mathbb{T})$, on the compact operators $\mathcal{K}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$, and lastly on irrational rotation C^*-algebras A_ϑ.

III.2. Ergodic theory of C^*-systems

We borrow the notation introduced in Section II.4. Let $(\mathfrak{A}, G, \alpha)$ be a C^*-system and $\varphi \in \mathcal{S}_G(\mathfrak{A})$. Recall that

$$\mathcal{H}_\varphi^G := \{\xi \in \mathcal{H}_\varphi \colon U_\varphi(g)\xi = \xi,\, g \in G\} = \bigcap_{g\in G} \ker(U_\varphi(g) - I)$$

is the Hilbert subspace of $U_\varphi(G)$-invariant vectors and $E_\varphi \colon \mathcal{H}_\varphi \twoheadrightarrow \mathcal{H}_\varphi^G$ is the orthogonal projection onto $\mathcal{H}_\varphi$. By Lemma 4.1 in [7] (p. 15), $\mathcal{H}_\varphi^G$ is the largest Hilbert subspace $\mathcal{K} \subseteq \mathcal{H}_\varphi$ s.t.

- $\xi_\varphi \in \mathcal{K}$
- for each $\xi \in \mathcal{K}$, $v_\varphi^\xi \colon a \mapsto \langle \pi_\varphi(a)\xi, \xi\rangle_{\mathcal{H}_\varphi} \in \mathcal{S}_G(\mathfrak{A})$.

In particular, $\mathcal{H}_\varphi = \mathcal{H}_\varphi^G \oplus (\mathcal{H}_\varphi^G)^\perp$ (where both $\mathcal{H}_\varphi^G, (\mathcal{H}_\varphi^G)^\perp$ are $U_\varphi(G)$-invariant) and $E_\varphi U_\varphi(g) = U_\varphi(g)E_\varphi = E_\varphi$ for each $g \in G$. The *compression* (or *corner*) mapping

$$\begin{aligned} E_\varphi \mathcal{B}(\mathcal{H}_\varphi)E_\varphi &\to \mathcal{B}(\mathcal{H}_\varphi^G) \\ E_\varphi X E_\varphi &\mapsto X^G := E_\varphi X|_{\mathcal{H}_\varphi^G} \end{aligned}$$

is a *-isomorphism of C^*-algebras, so that $E_\varphi \pi_\varphi(\mathfrak{A})E_\varphi$ can be identified with an operator system $\pi_\varphi(\mathfrak{A})^G$ acting upon $\mathcal{H}_\varphi^G$ (in general, it is not a C^*-subalgebra since $E_\varphi \notin \pi_\varphi(\mathfrak{A})'$). Surprisingly, by Corollary 2 in [22] (p. 422), $E_\varphi \pi_\varphi(\mathfrak{A})'' E_\varphi$ is a von Neumann algebra.

Extremality in $\mathcal{S}_G(\mathfrak{A})$ can be usefully linked to several cluster properties, as well as to the form of the GNS invariant Hilbert subspace, as summarized in the following well-known result.

Theorem III.2.1 (Cluster properties in $\mathcal{S}_G(\mathfrak{A})$)
Let $(\mathfrak{A}, G, \mathfrak{A})$ be a C^*-system and $\varphi \in \mathcal{S}_G(\mathfrak{A})$. Consider the list of properties below:

(1) φ is *strongly clustering*: there exists $(g_n)_{n\in\mathbb{N}} \subset G$ s.t. for each pair $a, b \in \mathfrak{A}$

$$\lim_{n\to+\infty} |\varphi(g_n(a)b) - \varphi(a)\varphi(b)| = 0$$

(2) φ is *weakly clustering*: for each pair $a, b \in \mathfrak{A}$,

$$\inf_{x \in \mathrm{co}(G \cdot a)} |\varphi(xb) - \varphi(a)\varphi(b)| = 0$$

(3) for each $a \in \mathfrak{A}$, there exists a net $\{a_\lambda\}_{\lambda \in \Lambda} \subseteq \mathrm{co}(G \cdot a)$ s.t.

$$\lim_{\lambda \in \Lambda} |\varphi(g(a_\lambda)b) - \varphi(a)\varphi(b)| = 0, \quad b \in \mathfrak{A},\ g \in G$$

(4) φ is 3-*weakly clustering*: for each triplet $a, b, B \in \mathfrak{A}$, $\inf_{x \in \mathrm{co}(G \cdot a)} |\varphi(Bxb) - \varphi(a)\varphi(Bb)| = 0$

(5) for each $a \in \mathfrak{A}$, there exists a net $\{a_\lambda\}_{\lambda \in \Lambda} \subseteq \mathrm{co}(G \cdot a)$ s.t.

$$\lim_{\lambda \in \Lambda} |\varphi(Bg(a_\lambda)b) - \varphi(a)\varphi(Bb)| = 0, \quad b, B \in \mathfrak{A},\ g \in G$$

(6) $\mathcal{H}_\varphi^G = \mathbb{C}\xi_\varphi$

(7) $\pi_\varphi(\mathfrak{A})' \cap U_\varphi(G)' = \mathbb{C}I_{\mathcal{H}_\varphi}$

(8) $\varphi \in \mathcal{E}_G(\mathfrak{A})$

The following implication scheme holds:

$$\begin{array}{ccccc}
(1) & & & & (4) \Leftarrow (5) \\
\Downarrow & & & & \Downarrow \\
(2) & \Longleftrightarrow & (3) & \Longleftrightarrow & (6) \\
 & & & & \Downarrow \\
 & & & & (7) \Longleftrightarrow (8)
\end{array}$$

In case $\xi_\varphi \in \mathcal{H}_\varphi$ is separating for $\pi_\varphi(\mathfrak{A})''$ (i.e. cyclic for $\pi_\varphi(\mathfrak{A})'$), then (4) $\Longleftrightarrow$ (5) $\Longleftrightarrow$ (6).

Proof.
(1)$\Rightarrow$(2) is apparent. The equivalences (2) $\Longleftrightarrow$ (3) $\Longleftrightarrow$ (6) are the content of Theorem 4.3.22 in [86] (p. 398). The implications (6) $\Rightarrow$ (7) $\Longleftrightarrow$ (8) are the content of Proposition 3.1.10 in [99] (p. 126). The implications (5) $\Rightarrow$ (4) $\Rightarrow$ (6) and the last assertion are the content of Proposition 4.3.23 in [86] (p. 399). □

Remark III.2.2
Just to point out, by taking the adjoints, (1), (2), (3) in Theorem III.2.1 are equivalent, respectively, to

(1') there exists $(g_n)_{n \in \mathbb{N}} \subset G$ s.t. for each pair $a, b \in \mathfrak{A}$ $\lim_{n \to +\infty} |\varphi(bg_n(a)) - \varphi(a)\varphi(b)| = 0$

(2') for each pair $a, b \in \mathfrak{A}$, $\inf_{x \in \mathrm{co}(G \cdot a)} |\varphi(bx) - \varphi(a)\varphi(b)| = 0$

(3') for each $a \in \mathfrak{A}$, there exists a net $\{a_\lambda\}_{\lambda \in \Lambda} \subseteq \mathrm{co}(G \cdot a)$ s.t. $\lim_{\lambda \in \Lambda} |\varphi(bg(a_\lambda)) - \varphi(a)\varphi(b)| = 0$ for every $b \in \mathfrak{A}$, $g \in G$.

A fundamental notion in the ergodic theory of C^*-systems, originally introduced by Lanford and Ruelle in [55], is G-abelianness of an invariant state. A state $\varphi \in \mathcal{S}_G(\mathfrak{A})$ is said to be *G-abelian* if the operator system $\pi_\varphi(\mathfrak{A})^G \subseteq \mathcal{B}(\mathcal{H}_\varphi^G)$ consists of mutually commuting operators (we will also say that $\pi_\varphi(\mathfrak{A})^G$ is *abelian*, even if it is not an algebra). The C^*-system $(\mathfrak{A}, G, \alpha)$ is *G-abelian* if $\mathcal{S}_G(\mathfrak{A})$ consists of G-abelian states. The relevance of G-abelian C^*-systems is motivated by the fact that they suffice (actually, they are even necessary as we will see soon) to make the convex, weakly-* compact family of G-invariant states a *simplex*, in the sense of Choquet. Generally speaking, given a locally convex linear space E (over $\mathbb{R}$ or $\mathbb{C}$), a convex set C lying in a hyperplane $H \subset E$ s.t. $0 \notin H$ is a *Choquet simplex* if its cone $\widetilde{C} := \{tx : x \in C, t \geq 0\}$ is a $\leq_{\widetilde{C}}$-lattice (i.e. a $\leq_{\widetilde{C}}$-poset which is closed under l.u.b. and g.l.b. of any *finite* set of elements, where $y \leq_{\widetilde{C}} x$ if and only if $x - y \in \widetilde{C}$). Easy examples of Choquet simplices are, of course, the classical simplices in finite-dimensional linear spaces, but also the space $\mathcal{M}_1(X)$ of probability measures on a compact Hausdorff space X (i.e. the positive Borel measures μ on X, which are inner-outer regular and s.t. $\mu(X) = 1$) and the space of tracial states on an arbitrary C^*-algebra. For a comprehensive discussion on Choquet theory, see [98]. Here, we limit ourselves to give a standard characterization of Choquet simplices, which encaptures their fundamental importance in integral representation theory.

Theorem III.2.3 (Characterization of Choquet simplices)
Let C be a convex, compact set in a locally convex linear space. The following properties are equivalent:

(1) C is a Choquet simplex

(2) for each $x \in C$, there exists a *unique* $\mu \in \mathcal{M}_1(C)$ s.t.

(i) $\displaystyle\int_C f \, \mathrm{d}\mu = f(x)$ for each affine $f \in \mathcal{C}(C, \mathbb{R})$

(ii) if $\nu \in \mathcal{M}_1(C)$ satisfies $\displaystyle\int_C f \, \mathrm{d}\nu \geq \int_C f \, \mathrm{d}\mu$ for each convex $f \in \mathcal{C}(C, \mathbb{R})$, then $\nu = \mu$

(3) for each convex $f \in \mathcal{C}(C, \mathbb{R})$, $u_f : x \mapsto \inf \Big\{ g(x) : g \geq f, -g \in \mathcal{C}(C, \mathbb{R}) \text{ convex} \Big\}$ is affine

(4) there is an affine assignment $\begin{matrix} C \to \mathcal{M}_1(C) \\ x \mapsto \mu(x) \end{matrix}$ where $x, \mu(x)$ satisfy (i).

Proof.
See Theorem 4.1.15 (p. 335) and Corollary 4.1.17 (p. 337) in [86]. □

If $x \in C$, $\mu \in \mathcal{M}_1(C)$ satisfy point (i) in Theorem III.2.3, we say that μ has x as *barycenter*. It can be proved that it is unique (see Proposition 4.1.1 in [86], p. 323). Moreover, in view of the Riesz-Markov-Kakutani theorem, $\mathcal{M}_1(C) \cong \mathcal{S}(\mathcal{C}(C))$ allowing us to write "$\mu(f) = f(x)$" for each affine $f \in \mathcal{C}(C, \mathbb{R})$. If $\mu \in \mathcal{M}_1(C)$ satisfies point (ii) in Theorem III.2.3, we say that μ is $\prec$-maximal (or simply, maximal) in $\mathcal{M}_1(C)$, where $\prec$ is the partial ordering relation on $\mathcal{M}_1(C)$ s.t. $\nu_1 \prec \nu_2$ ($\nu_i \in \mathcal{M}_1(C)$, $i = 1, 2$) iff $(\nu_2 - \nu_1)(f) \geq 0$ for every convex $f \in \mathcal{C}(C, \mathbb{R})$. Lastly, u_f in point (3) of Theorem III.2.3 is said to be the *upper envelope* of f, a concave and upper semicontinuous function on C (see Proposition 4.1.6 in [86], p. 327).

It is worth noticing that the real essence of Choquet simplices does not rely on the *existence* of a barycentric decomposition of their points by maximal probability measures: this is true for *any* convex compact set in a locally convex linear space (it is known as the Choquet-Bishop-de

Leeuw theorem) and can be even relaxed to hold for every affine upper semicontinuous (not necessarily continuous) function on C (see Corollary 4.1.18 in [86], p. 338). Instead, it is the *uniqueness* of this decomposition that characterizes Choquet simplices among all the convex compact subsets of a locally convex linear space, and this result is known as the Choquet-Meyer theorem. It will allow us to uniquely represent each state of a G-abelian C^*-system $(\mathfrak{A}, G, \alpha)$ as the barycentric point of the associated maximal measure. We can say even more about maximal probability measures on general convex compact sets. For that, let $\mathcal{B}(C)$ the Borel σ-algebra of C (the one generated by its topology) and $\mathcal{B}_0(C) \subseteq \mathcal{B}(C)$ its Baire σ-subalgebra (the one generated by the closed G_δ-sets in C).

Proposition III.2.4
Let C be a convex, compact set in a locally convex linear space, $\mu \in \mathcal{M}_1(C)$ be $\prec$-maximal. Then,

(1) μ is *pseudo-supported* by $\mathrm{Ext}(C)$: every $B \in \mathcal{B}_0(C)$ s.t. $B \subseteq \mathrm{Ext}(C)^c$ is μ-null, or equivalently $\mu(B) = 1$ for every $B \in \mathcal{B}_0(C)$ s.t. $B \supseteq \mathrm{Ext}(C)$.

(2) if C is metrizable, then $\mathcal{B}_0(C) = \mathcal{B}(C)$ and $\mathrm{Ext}(C)$ is a G_δ-set. In particular, μ is *supported* by $\mathrm{Ext}(C)$: every $B \in \mathcal{B}(C)$ s.t. $B \subseteq \mathrm{Ext}(C)^c$ is μ-null, or equivalently $\mu(\mathrm{Ext}(C)) = 1$.

Proof.
See Theorem 4.1.11 in [86] (p. 331). □

Coming back to our initial setting, G-abelian C^*-systems are precisely the ones for which the space $\mathcal{S}_G(\mathfrak{A})$ of G-invariant states is a Choquet simplex, hence admitting an unique ergodic decomposition of its elements via maximal probability measures. We formalize this statement in the following two results.

Theorem III.2.5 (Characterization of G-abelian C^*-systems [Batty])
Let $(\mathfrak{A}, G, \alpha)$ be a C^*-system. Consider the list of properties below:

(1) $(\mathfrak{A}, G, \alpha)$ is G-abelian

(2) the W^*-algebra $\pi_\varphi(\mathfrak{A})' \cap U_\varphi(G)'$ is abelian for each $\varphi \in \mathcal{S}_G(\mathfrak{A})$

(3) $\varphi \in \mathcal{S}_G(\mathfrak{A})$ is a Choquet simplex

(4) if $\varphi \in \mathcal{S}_G(\mathfrak{A})$ is s.t. $\pi_\varphi(\mathfrak{A})' \cap U_\varphi(G)'$ is a factor, then $\varphi \in \mathcal{E}_G(\mathfrak{A})$

(5) two distinct $\varphi, \psi \in \mathcal{E}_G(\mathfrak{A})$ are covariantly inequivalent (i.e. $(\mathcal{H}_\varphi, \pi_\varphi, \xi_\varphi) \not\sim (\mathcal{H}_\psi, \pi_\psi, \xi_\psi)$)

(6) $\mathcal{S}_G(\mathfrak{A})$ has the 1-ball property ($[\varphi, \psi]$ is a face for every $\varphi, \psi \in \mathcal{E}_G(\mathfrak{A})$)

(7) if $\varphi \in \mathcal{E}_G(\mathfrak{A})$, then φ is weakly clustering

The following implication scheme holds:

$$\begin{array}{ccccc} (1) & \Longleftrightarrow & (2) & \Longleftrightarrow & (3) \\ & & & & \Downarrow \\ & & & & (4) \\ & & & & \Downarrow \\ & & & & (5) \Longleftrightarrow (6) \Longleftrightarrow (7) \end{array}$$

In case either $\mathfrak{A}$ is *separable* or G is *σ-compact*, all seven conditions are equivalent.

Proof.
See Corollary 4.4 (p. 18) in [7] and the observations below. □

For the following corollary, recall that for any (unital) C^*-algebra $\mathfrak{A}$, the evaluation map establishes an homeomorphism between $\mathcal{S}(\mathfrak{A})$, endowed with the weak-* topology, and the *spectrum* (character space/maximal ideal space) $\Omega_{\mathcal{C}(\mathcal{S}(\mathfrak{A}))}$ of $\mathcal{C}(\mathcal{S}(\mathfrak{A}))$, endowed with the topology of pointwise convergence:

$$\begin{aligned} \mathrm{ev}\colon \mathcal{S}(\mathfrak{A}) &\xrightarrow{\cong} \Omega_{\mathcal{C}(\mathcal{S}(\mathfrak{A}))} \\ \varphi &\mapsto [\mathrm{ev}_\varphi\colon f \mapsto f(\varphi)] \end{aligned}$$

On the other hand, the Riesz-Markov-Kakutani theorem represents each member of $\Omega_{\mathcal{C}(\mathcal{S}(\mathfrak{A}))}$, i.e. ev_φ for some $\varphi \in \mathcal{S}(\mathfrak{A})$, as the Dirac probability measure on $\mathcal{S}(\mathfrak{A})$ centered at φ, that is the unique $\mu_\varphi \in \mathcal{M}_1(\mathcal{S}(\mathfrak{A}))$ satisfying

$$f(\varphi) = \int\limits_{\mathcal{S}(\mathfrak{A})} f(\omega)\mathrm{d}\mu_\varphi(\omega), \quad f \in \mathcal{C}(\mathcal{S}(\mathfrak{A}))\,. \tag{III.2}$$

Now, since $\mathcal{S}(\mathfrak{A})$ is point-separating, there exists a contractive linear injection

$$\begin{aligned} \sigma\colon \mathfrak{A} &\hookrightarrow \mathcal{C}(\mathcal{S}(\mathfrak{A})) \\ a &\mapsto [\widehat{a}\colon \varphi \mapsto \varphi(a)] \end{aligned}$$

whose restriction $\sigma|_{\mathfrak{A}_{\mathrm{sa}}}\colon \mathfrak{A}_{\mathrm{sa}} \hookrightarrow \mathcal{C}(\mathcal{S}(\mathfrak{A}), \mathbb{R})$ to the Jordan subalgebra of selfadjoint elements of $\mathfrak{A}$ is isometric. Writing Equation III.2 specifically for $\widehat{a} \in \mathcal{C}(\mathcal{S}(\mathfrak{A}))$ ($a \in \mathfrak{A}$),

$$\varphi(a) = \int\limits_{\mathcal{S}(\mathfrak{A})} \omega(a)\mathrm{d}\mu_\varphi(\omega), \quad \varphi \in \mathcal{S}(\mathfrak{A}) \tag{III.3}$$

Lastly, by the Hahn-Banach theorem,

$$\overline{\sigma(\mathfrak{A})}^{\mathcal{C}(\mathcal{S}(\mathfrak{A}))} = \overline{\{\widehat{a} \in \mathcal{C}(\mathcal{S}(\mathfrak{A}))\colon a \in \mathfrak{A}\}}^{\mathcal{C}(\mathcal{S}(\mathfrak{A}))} = \{f \in \mathcal{C}(\mathcal{S}(\mathfrak{A}))\colon f \text{ affine}\}$$

and hence Equation III.3 simply tells us that φ is the barycenter of μ_φ.
This fact can be generalized to any non-empty, convex and weakly-* closed subset X of $\mathcal{S}(\mathfrak{A})$:

$$\overline{\sigma(\mathfrak{A})|_X}^{\mathcal{C}(X)} = \overline{\{\widehat{a}|_X\colon a \in \mathfrak{A}\}}^{\mathcal{C}(X)} = \{f \in \mathcal{C}(X)\colon f \text{ affine}\}\,.$$

In particular, if $(\mathfrak{A}, G, \alpha)$ is a C^*-system and $\varphi \in \mathcal{S}_G(\mathfrak{A})$, $\mu_\varphi \in \mathcal{M}_1(\mathcal{S}_G(\mathfrak{A}))$ has φ as barycenter iff $\varphi(a) = \int\limits_{\mathcal{S}_G(\mathfrak{A})} \omega(a)\mathrm{d}\mu_\varphi(\omega)$ for every $a \in \mathfrak{A}$. We can then write

Corollary III.2.6
Let $(\mathfrak{A}, G, \alpha)$ be a C^*-system. Then, the following are equivalent:

- $(\mathfrak{A}, G, \alpha)$ is G-abelian
- for every $\varphi \in \mathcal{S}_G(\mathfrak{A})$, there exists a unique $\prec$-maximal $\mu_\varphi \in \mathcal{M}_1(\mathcal{S}_G(\mathfrak{A}))$ such that

$$\varphi(a) = \int\limits_{\mathcal{S}_G(\mathfrak{A})} \omega(a)\mathrm{d}\mu_\varphi(\omega), \quad a \in \mathfrak{A}\,.$$

Proof.
This is just a restatement of the equivalence (1) $\iff$ (3) in Theorem III.2.5 together with Theorem III.2.3. □

A Choquet simplex C in a locally convex linear space E is a *Bauer simplex* if its extremal boundary $\mathrm{Ext}(C)$ is closed. This is clearly the case for the Choquet simplex $\mathcal{S}(\mathfrak{A})$ of the states of a unital abelian C^*-algebra $\mathfrak{A}$, where $\mathrm{Ext}(\mathcal{S}(\mathfrak{A})) = \mathcal{P}(\mathfrak{A})$ is the family of the *pure* states, coinciding with its spectrum $\Omega_{\mathfrak{A}}$. In Theorem 3.9 of [77] (p. 16), Størmer gives a necessary and sufficient condition for the weak-* closure of $\mathcal{E}_G(\mathfrak{A})$ in $\mathcal{S}_G(\mathfrak{A})$, given a G-abelian C^*-system $(\mathfrak{A}, G, \alpha)$. We shall evoke this result later on.

Theorem III.2.7
Let $(\mathfrak{A}, G, \alpha)$ be a G-abelian C^*-system. Then, $\mathcal{E}_G(\mathfrak{A})$ is weakly-* closed if and only if there exists a G-invariant, densely ranged p.u. map of C^*-algebras $T\colon \mathfrak{A} \to \mathcal{C}(\mathcal{E}_G(\mathfrak{A}))$ s.t. the transpose $T^{\mathrm{t}}\colon \mathcal{M}_1(\mathcal{E}_G(\mathfrak{A})) \to \mathcal{S}_G(\mathfrak{A})$ is an affine homeomorphism.

Various notions of *asymptotic abelianness* have been introduced over time to guarantee that $\mathcal{S}_G(\mathfrak{A})$ forms a Choquet simplex. One of them is in [77] (p. 17), where a C^*-system $(\mathfrak{A}, G, \alpha)$ is said to be *asymptotic abelian* if for each fixed $a \in \mathfrak{A}_{\mathrm{sa}}$ there exists a sequence $\{g_{n,a}\}_{n\in\mathbb{N}} \subseteq G$ for which $\lim_{n\to+\infty} \|[g_{n,a}(a), b]\| = 0$, $b \in \mathfrak{A}$ (actually, here Størmer does not even assume the strong continuity of the G-action). The asymptotic abelianness property here introduced guarantees that G acts as a *large group of automorphisms* on $\mathfrak{A}$ (we will simply say that G acts largely), i.e. for every $\varphi \in \mathcal{S}_G(\mathfrak{A})$ and $a \in \mathfrak{A}_{\mathrm{sa}}$,

$$\overline{\mathrm{co}}^w(\pi_\varphi(G \cdot a)) \cap \pi_\varphi(\mathfrak{A})' \neq \varnothing$$

where $\overline{\cdot}^w$ denotes the closure in the weak operator topology of $\mathcal{B}(\mathcal{H}_\varphi)$. Precisely, Størmer proves the following

Theorem III.2.8
Let $(\mathfrak{A}, G, \alpha)$ be a C^*-system. Then, G acts largely if and only if for each $\varphi \in \mathcal{S}_G(\mathfrak{A})$, $c \in \mathfrak{A}$ and each finite family $\{a, b_i\}_{i=1}^n \subset \mathfrak{A}_{\mathrm{sa}}$,

$$\inf_{x\in\mathrm{co}(G\cdot a)} |\varphi(c^*[x, b_i]c)| = 0, \quad i = 1, \dots, n.$$

Proof.
This is Theorem 3.5 in [77] (p. 13). □

Moreover, point (4) in Theorem 3.1 of [77] (p. 9) guarantees that asymptotic abelianness implies G-abelianness of the C^*-system, or equivalently that $\mathcal{S}_G(\mathfrak{A})$ is a Choquet simplex.
We would like to conclude the present section with two enlightening observations. The former, again due to Batty ([7], p. 16-17), works well in the setting of *discrete* C^*-systems, i.e. systems where the acting group G is discrete. The latter can be found in [18] and concerns the case when a discrete group G is obtained by the direct limit of an ascending chain of finite subgroups. The finitary symmetric group $\mathfrak{S}_J$ (J being any set) fits both situations.
Let us start from Batty's observation, hence suppose that G is a discrete group acting on a C^*-algebra $\mathfrak{A}$ via α. Then, one can easily build the *full* crossed product (also referred to as the *universal*, or *maximal* crossed product) $\mathfrak{A} \rtimes_{\alpha,\mathrm{f}} G$ as the completion of the group convolution *-algebra $\mathcal{C}_c(\mathfrak{A}, G, \alpha)$ w.r.t. to the universal norm $\|\cdot\|_{\alpha,\mathrm{f}} := \sup_{(\pi,U)\in\mathsf{Cov}(\mathfrak{A},G,\alpha)} \|\rho_{(\pi,U)}(\cdot)\|$, where $\rho_{(\pi,U)}(f)\xi := \sum_{g\in G} \pi(f_g)U_g\xi \in \mathsf{Rep}(\mathcal{C}_c(G, \mathfrak{A}, \alpha))$ for $f \in \mathcal{C}(\mathfrak{A}, G, \alpha)$ is the integrated form of the

covariant representation (π, U) on $\mathcal{H}_{(\pi,U)}$. Both $\mathfrak{A}$ and G canonically embed into $\mathfrak{A} \rtimes_{\alpha,\mathrm{f}} G$: in particular, let $g \mapsto u_g$ be the embedding of G into $\mathfrak{A} \rtimes_{\alpha,\mathrm{f}} G$. The universal property of $\mathfrak{A} \rtimes_{\alpha\mathrm{f}} G$ consists in the fact that the assignment $(\pi, U) \mapsto \rho_{(\pi,U)}$ establishes (up to unitary equivalence of representations) a bijection between $\mathsf{Cov}(\mathfrak{A}, G, \alpha)$ and non-degenerate representations of $\mathfrak{A} \rtimes_{\alpha,\mathrm{f}} G$. In addition, if

$$B^1_+(\mathfrak{A}\rtimes_{\alpha,\mathrm{f}}G) := \left\{\Phi \in \ell^\infty(G, \mathfrak{A}^*) \colon \Phi(e) \in \mathcal{S}(\mathfrak{A}), \sum_{i,j=1}^{n} \Phi(g_i^{-1}g_j)\left(\alpha_{g_i}^{-1}(a_i^* a_j)\right) \geq 0, a_i \in \mathfrak{A}, g_i \in G, n \in \mathbb{N}\right\}$$

is the family of (*normalized*) *α-positive definite* functions in $\ell^\infty(G, \mathfrak{A}^*)$, then the map

$$\begin{aligned} \Phi \colon \mathcal{S}(\mathfrak{A} \rtimes_{\alpha,\mathrm{f}} G) &\to B^1_+(\mathfrak{A} \rtimes_{\alpha,\mathrm{f}} G) \\ \varphi &\mapsto \left[\Phi_\varphi \colon g \mapsto \left[\Phi_\varphi(g) \colon a \mapsto \varphi(a u_g)\right]\right] \end{aligned}$$

is an affine homeomorphism, provided that $\mathcal{S}(\mathfrak{A} \rtimes_{\alpha,\mathrm{f}} G)$ and $B^1_+(G, \mathfrak{A}^*)$ are endowed with the weak-* and the pointwise weak-* convergence topologies, respectively (see Proposition 7.6.10, p. 330, in [97]). The following result can be found as Theorem 4.2 in [7] (p. 17). For that, recall that a convex subset $\mathcal{F} \subseteq \mathcal{S}(\mathfrak{A})$ is a *face* if whenever $\omega_1, \omega_2 \in \mathcal{S}(\mathfrak{A})$ satisfy $\mathcal{F} \cap (\omega_1, \omega_2) \neq \varnothing$, $\omega_1, \omega_2 \in \mathcal{F}$ (or equivalently, whenever $\varphi \in \mathcal{F}$, $\omega \in \mathcal{S}(\mathfrak{A})$ satisfy $\omega \leq t\varphi$ for some $t > 0$, $\omega \in \mathcal{F}$). Given a weakly-* closed face $\mathcal{F} \subseteq \mathcal{S}(\mathfrak{A})$, the generated cone $\widetilde{\mathcal{F}}$ is a weakly-* closed order-ideal in the cone $\mathfrak{A}^*_+$ of positive functionals.

Theorem III.2.9 (Batty, 1980)
Let $(\mathfrak{A}, G, \alpha)$ be a C^*-system, with G discrete. Then, $\mathcal{S}_G(\mathfrak{A})$ is affinely homeomorphic to the weakly-* closed face of the G-multiplication invariant states of $\mathfrak{A} \rtimes_{\alpha,\mathrm{f}} G$

$$\mathcal{F}_\rtimes := \{\varphi \in \mathcal{S}(\mathfrak{A} \rtimes_{\alpha,\mathrm{f}} G) \colon \varphi(u_g a u_h) = \varphi(a), \quad g, h \in G, a \in \mathfrak{A}\}$$

via the assignment $T \colon \begin{array}{l} \mathcal{F}_\rtimes \to \mathcal{S}_G(\mathfrak{A}) \\ \varphi \mapsto \Phi_\varphi(e) \end{array}$. Moreover, for every $\varphi \in \mathcal{F}_\rtimes$, the two GNS quadruplets

$$(\mathcal{H}_\varphi, \mathcal{H}_\varphi^{\mathcal{F}_\rtimes}, \pi_\varphi, \xi_\varphi) \qquad (\mathcal{H}_{T\varphi}, \mathcal{H}^G_{T\varphi}, \rho_{(\pi_{T\varphi}, U_{T\varphi})}, \xi_{T\varphi})$$

are unitarily equivalent, where $\mathcal{H}_\varphi^{\mathcal{F}_\rtimes} := \{\xi \in \mathcal{H}_\varphi \colon a \mapsto \langle \pi_\varphi(a)\xi, \xi\rangle_{\mathcal{H}_\varphi} \in \widetilde{\mathcal{F}_\rtimes}\}$.

Proof.
See Theorem 4.2 in [7] (p. 17). □

The second remark we want to make is about (discrete) groups G admitting an *ascending chain* of finite subgroups $\{G_\lambda\}_{\lambda\in\Lambda}$ with Λ directed family of indices (i.e. $\lambda \leq \lambda'$ implies $G_\lambda \leq G_{\lambda'}$) for which

$$G = \lim_{\longrightarrow \lambda} G_\lambda = \bigcup_{\lambda \in \Lambda} G_\lambda \tag{III.4}$$

the direct limit being taken w.r.t. to the group embeddings $\phi_{\lambda\lambda'} \colon G_\lambda \hookrightarrow G_{\lambda'}$, $\lambda \leq \lambda'$). In particular, G must be *locally finite* (every finitely generated subgroup is, in fact, finite) and *amenable* (it admits a finitely-additive and left-invariant probability measure). Given a unitary representation $U \colon G \to \mathcal{U}(\mathcal{H})$ of G on a Hilbert space $\mathcal{H}$, for every $\lambda \in \Lambda$ let $\mathcal{H}_\lambda := \{\xi \in \mathcal{H} \colon U_g \xi = \xi, \, g \in G_\lambda\}$ and $E_\lambda \in \mathcal{B}(\mathcal{H})$ the orthogonal projection onto $\mathcal{H}_\lambda$. Similarly, define $\mathcal{H}^G := \{\xi \in \mathcal{H} \colon U_g\xi = \xi, \, g \in G\}$ and $E \in \mathcal{B}(\mathcal{H})$ the orthogonal projection onto $\mathcal{H}^G$. Then, $E_\lambda = \dfrac{1}{|G_\lambda|} \displaystyle\sum_{g \in G_\lambda} U_g$ $(\lambda \in \Lambda)$ and $\{E_\lambda\}_{\lambda\in\Lambda} \subseteq \mathcal{B}(\mathcal{H})$ is a decreasing net, hence converging (in s.o.t.) to the orthogonal projection $P := s\text{-}\lim_\lambda E_\lambda \in \mathcal{B}(\mathcal{H})$ onto the closed subspace $\bigcap_{\lambda\in\Lambda} \mathcal{H}_\lambda = \{\xi \in \mathcal{H}\colon U_g\xi = \xi, g \in G_\lambda, \lambda \in \Lambda\}$. It is apparent that $\mathcal{H}^G \subseteq \bigcap_{\lambda\in\Lambda} \mathcal{H}_\lambda$, so that $E \leq P$. Actually, the equality holds, thanks to a version of the celebrated von Neumann Ergodic Theorem reported in Proposition 3.1 of [18] (p. 141). We can then write E as a Cesàro mean of the unitary group $\{U_g\}_{g\in G}$ i.e. $E = \mathcal{M}\{U_g\} := s\text{-}\lim_\lambda \frac{1}{|G_\lambda|}\sum_{g\in G_\lambda} U_g$.

More in general, we formally define

$$\mathcal{M}\{f(g)\} := \lim_\lambda \frac{1}{|G_\lambda|}\sum_{g\in G_\lambda} f(g)$$

as the *Cesàro mean* of the assignment $G \ni g \mapsto f(g) \in \mathcal{X}$, provided that the r.h.s. exists in the topology of a suitable topological space $\mathcal{X}$. For instance, if $(\mathfrak{A}, G, \alpha)$ is a C^*-system with G satisfying Equation III.4 and $\varphi \in \mathcal{S}_G(\mathfrak{A})$, then $\mathcal{M}\{\varphi(f(g))\}$ (for some map $f\colon G \to \mathfrak{A}$) will be always meant in the Euclidean topology on $\mathbb{C}$. For instance, in the analysis of the symmetric states on the CAR algebra, Crismale and Fidaleo introduce two properties in Theorem 4.1 of [18] (p. 143), the *asymptotic abelianness in average* and the *weak clustering in average*. An G-invariant state $\omega \in \mathcal{S}_G(\mathfrak{A})$ is said to be *asymptotically abelian in average* if $\mathcal{M}\{\omega(c[g(a), b]d)\} = 0$ for every $a, b, c, d \in \mathfrak{A}$ (or equivalently, $\mathcal{M}\{\omega(c[b, g(a)]d)\} = 0$ for $a, b, c, d \in \mathfrak{A}$). This property is introduced in . Though weaker than Størmer's asymptotic abelianness exposed above, this new property still suffices to make G act largely on $\mathfrak{A}$. Indeed, observe that for every $a \in \mathfrak{A}$, $\left\{\frac{1}{|G_\lambda|}\sum_{g\in G_\lambda} g(a)\right\}_{\lambda\in\Lambda}$ is a net lying in $\mathrm{co}(G\cdot a)$. Therefore, if $\omega \in \mathcal{S}_G(\mathfrak{A})$ is asymptotically abelian in average,

$$\lim_\lambda \omega\left(c^*\left[\frac{1}{|G_\lambda|}\sum_{g\in G_\lambda} g(a), b\right] c\right) = \mathcal{M}\{\omega(c^*[g(a), b]c)\} = 0$$

for every $a, b, c \in \mathfrak{A}$. By Theorem III.2.8, it follows that G acts largely, and consequently that $\mathcal{S}_G(\mathfrak{A})$ is a Choquet simplex. We will see in Subsection III.7.2 that this is the case for the C^*-system $(\mathfrak{A}, \mathfrak{S}, \alpha)$ where the finitary symmetric group $\mathfrak{S}$ acts on a Fermi (minimal) C^*-chain $\mathfrak{A} = \textcircled{\scriptsize F}_{n\in\mathbb{N}} \mathfrak{B}$ based on a fixed $\mathbb{Z}_2$-graded C^*-algebra $\mathfrak{B}$. On the contrary, the large G-action property will be not guaranteed in general when the grading of $\mathfrak{B}$ comes from the Klein 4-group $K_4 = \mathbb{Z}_2 \times \mathbb{Z}_2$, as it will be shown in Section III.8. Nonetheless, we will still be able to ensure the $\mathfrak{S}$-abelianness even in this case.

In much the same way, $\omega \in \mathcal{S}_G(\mathfrak{A})$ is said to be *weakly clustering in average* if $\mathcal{M}\{\omega(g(a)b)\} = \omega(a)\omega(b)$ for every pair $a, b \in \mathfrak{A}$ (or equivalently, $\mathcal{M}\{\omega(ag(b))\} = \omega(a)\omega(b)$ for $a, b \in \mathfrak{A}$). This property realizes point (3) in Theorem III.2.1, equivalent to point (2) (weak clustering property of ω) and (6) ($\mathcal{H}_\omega^G = \mathbb{C}\xi_\omega$) of the same theorem. In particular, it implies ergodicity: $\omega \in \mathcal{E}_G(\mathfrak{A})$.

III.3. The infinite twisted C^*-tensor product

The present section is devoted to the construction of the C^*-inductive limit of a direct system consisting of (finitely many) twisted minimal C^*-tensor products, based on a fixed C^*-system $(\mathfrak{B}, G, \beta)$, with G compact abelian group acting on the unital C^*-algebra $\mathfrak{B}$. Henceforward, $\mathbb{N}_0 := \{0, 1, 2, \dots\}$ and $\mathbb{N} := \{1, 2, \dots\}$. The first step of our iterative construction will be the

C^*-system $(\mathfrak{A}_2, G, \delta^{(\beta)})$, where $\mathfrak{A}_2 := \mathfrak{B} \textcircled{u} \mathfrak{B}$ is the completion of the algebraic twisted tensor product $\mathfrak{B}_o \textcircled{u} \mathfrak{B}_o$ w.r.t the min-norm[1] and

$$\delta_g^{(\beta)}(x) := (\beta_g \textcircled{u} \beta_g)(x)\,, \quad x \in \mathfrak{A}_2,\ g \in G$$

is the *diagonal action* of G on $\mathfrak{A}_2$. More generally, every pair of C^*-systems $(\mathfrak{B}, G, \beta)$, $(\mathfrak{C}, G, \gamma)$, together with a bicharacter u on $\widehat{G}$, induces a new system $(\mathfrak{B} \textcircled{u} \mathfrak{C}, G, \delta^{(\beta,\gamma)})$ where G acts diagonally on $\mathfrak{B} \textcircled{u} \mathfrak{C}$ via $\delta^{(\beta,\gamma)}$:

$$\delta_g^{(\beta,\gamma)}(x) := (\beta_g \textcircled{u} \gamma_g)(x), \quad x \in \mathfrak{B} \textcircled{u} \mathfrak{C},\ g \in G\,.$$

Clearly, $\mathfrak{B}_o \textcircled{u} \mathfrak{C}_o \subseteq (\mathfrak{B} \textcircled{u} \mathfrak{C})_o$ but the inclusion may well be proper. For instance, the data $(\mathcal{C}(\mathbb{T}), \mathbb{Z}_2, r_\pi, u_{\mathrm{F}})$ suffice to produce the *π-rotation algebra* $(\mathcal{C}(\mathbb{T}) \textcircled{F} \mathcal{C}(\mathbb{T}), \mathbb{Z}_2, r_\pi \textcircled{F} r_\pi)$, for which given any $A, B \in \mathbb{C}$

$$A\cos(u) \textcircled{F} \cos(v) + B\sin(u) \textcircled{F} \sin(v) \in \left(\mathcal{C}(\mathbb{T}) \textcircled{F} \mathcal{C}(\mathbb{T})\right)_o \setminus \mathcal{C}(\mathbb{T})_o \textcircled{F} \mathcal{C}(\mathbb{T})_o$$

where $u, v \in \mathcal{C}(\mathbb{T})$ are the unitary generators of the first and second copy of $\mathcal{C}(\mathbb{T})$ respectively, and

$$\begin{cases} \cos(u) := u\text{-}\lim\limits_{n\to\infty} \sum\limits_{0\le k\le n} (-1)^n \dfrac{u^{2n}}{(2n)!} \\ \sin(u) := u\text{-}\lim\limits_{n\to\infty} \sum\limits_{0\le k\le n} (-1)^n \dfrac{u^{2n+1}}{(2n+1)!} \end{cases}$$

Nonetheless, G still acts diagonally on $\mathfrak{B}_o \textcircled{u} \mathfrak{C}_o$, so that $(\mathfrak{B}_o \textcircled{u} \mathfrak{C}_o, \|\ \|_{\min})$ is well a $\widehat{G}$-graded pre-C^*-algebra, obviously dense in $\mathfrak{B} \textcircled{u} \mathfrak{C}$. As a result, it makes sense to study $(\mathfrak{B}_o \textcircled{u} \mathfrak{C}_o) \textcircled{u} \mathfrak{D}_o$ for any other C^*-system $(\mathfrak{D}, G, \lambda)$. The following lemma asserts that the minimal twisted tensor product is *associative* in this particular case, thus giving a non-ambiguous meaning to expressions like $\mathfrak{A}_3 := \mathfrak{B} \textcircled{u} \mathfrak{B} \textcircled{u} \mathfrak{B}$, and more in general to the product $\mathfrak{A}_n$ of n copies of $\mathfrak{B}$.

Lemma III.3.1
With the notation above, $(\mathfrak{B}_o \textcircled{u} \mathfrak{C}_o) \textcircled{u} \mathfrak{D}_o \cong \mathfrak{B}_o \textcircled{u} (\mathfrak{C}_o \textcircled{u} \mathfrak{D}_o)$ as involutive algebras. The isomorphism extends to the minimal completions so that $(\mathfrak{B} \textcircled{u} \mathfrak{C}) \textcircled{u} \mathfrak{D} \cong \mathfrak{B} \textcircled{u} (\mathfrak{C} \textcircled{u} \mathfrak{D})$.

Proof.
By associativity of the tensor product $\odot$, it is clear that $(\mathfrak{B}_o \textcircled{u} \mathfrak{C}_o) \textcircled{u} \mathfrak{D}_o \cong \mathfrak{B}_o \textcircled{u} (\mathfrak{C}_o \textcircled{u} \mathfrak{D}_o)$ as $\mathbb{C}$-linear spaces. Let $*_L$ and $*_R$ be the adjoint operations on the involutive algebras $(\mathfrak{B}_o \textcircled{u} \mathfrak{C}_o) \textcircled{u} \mathfrak{D}_o$ and $\mathfrak{B}_o \textcircled{u} (\mathfrak{C}_o \textcircled{u} \mathfrak{D}_o)$, respectively. Similarly, let $\cdot_L$ and $\cdot_R$ be their respective products. For homogeneous $x_1 \in \mathfrak{B}_o$, $x_2 \in \mathfrak{C}_o$ and $x_3 \in \mathfrak{D}_o$ of degree $\sigma_i \in \widehat{G}$ $(i = 1, 2, 3)$ respectively, by recalling that $\partial(x_i \textcircled{u} x_j) = \sigma_i\sigma_j \in \widehat{G}$, we get

$$\begin{aligned} ((x_1 \textcircled{u} x_2) \textcircled{u} x_3)^{*L} &= \overline{u(\sigma_1\sigma_2, \sigma_3)}(x_1 \textcircled{u} x_2)^* \textcircled{u} x_3^* = \\ &= \overline{u(\sigma_1\sigma_2, \sigma_3)}\ \overline{u(\sigma_1, \sigma_2)}(x_1^* \textcircled{u} x_2^*) \textcircled{u} x_3^* = \\ &= \overline{u(\sigma_1, \sigma_2\sigma_3)}\ \overline{u(\sigma_2, \sigma_3)} x_1^* \textcircled{u} (x_2^* \textcircled{u} x_3^*) = \\ &= \overline{u(\sigma_1, \sigma_2\sigma_3)} x_1^* \textcircled{u} (x_2 \textcircled{u} x_3)^* = (x_1 \textcircled{u} (x_2 \textcircled{u} x_3))^{*R} \end{aligned}$$

where we have used the associativity of $\odot$ on the elementary tensors. If $X_1 \in \mathfrak{B}_o$, $X_2 \in \mathfrak{C}_o$ and $X_3 \in \mathfrak{D}_o$ are other three homogeneous elements of degree $\tau_i \in \widehat{G}$ $(i = 1, 2, 3)$, again using the associativity of $\odot$,

[1] Dealing with minimal C^*-tensor completions only, from now on we will always omit the subscript "min".

$$
\begin{aligned}
&((x_1 \mathbin{\textcircled{u}} x_2) \mathbin{\textcircled{u}} x_3) \cdot_L ((X_1 \mathbin{\textcircled{u}} X_2) \mathbin{\textcircled{u}} X_3) = \\
&\qquad = \overline{u(\tau_1\tau_2, \sigma_3)}\, ((x_1 \mathbin{\textcircled{u}} x_2)(X_1 \mathbin{\textcircled{u}} X_2)) \mathbin{\textcircled{u}} x_3X_3 = \\
&\qquad = \overline{u(\tau_1\tau_2, \sigma_3)}\ \overline{u(\tau_1, \sigma_2)}(x_1X_1 \mathbin{\textcircled{u}} x_2X_2) \mathbin{\textcircled{u}} x_3X_3 = \\
&\qquad = \overline{u(\tau_1, \sigma_2\sigma_3)}\ \overline{u(\tau_2, \sigma_3)} x_1X_1 \mathbin{\textcircled{u}} (x_2X_2 \mathbin{\textcircled{u}} x_3X_3) = \\
&\qquad = \overline{u(\tau_1, \sigma_2\sigma_3)} x_1X_1 \mathbin{\textcircled{u}} ((x_2 \mathbin{\textcircled{u}} x_3)(X_2 \mathbin{\textcircled{u}} X_3)) = \\
&\qquad\qquad = (x_1 \mathbin{\textcircled{u}} (x_2 \mathbin{\textcircled{u}} x_3)) \cdot_R (X_1 \mathbin{\textcircled{u}} (X_2 \mathbin{\textcircled{u}} X_3))\,.
\end{aligned}
$$

By extending the equality to all the finite $\mathbb{C}$-linear combinations of tensor products of homogeneous elements, it straightforwardly results that $(\mathfrak{B}_o \mathbin{\textcircled{u}} \mathfrak{C}_o) \mathbin{\textcircled{u}} \mathfrak{D}_o \cong \mathfrak{B}_o \mathbin{\textcircled{u}} (\mathfrak{C}_o \mathbin{\textcircled{u}} \mathfrak{D}_o)$ as involutive algebras, and hence $\overline{(\mathfrak{B}_o \mathbin{\textcircled{u}} \mathfrak{C}_o) \mathbin{\textcircled{u}} \mathfrak{D}_o}^{\min} \cong \overline{\mathfrak{B}_o \mathbin{\textcircled{u}} (\mathfrak{C}_o \mathbin{\textcircled{u}} \mathfrak{D}_o)}^{\min}$. Since the min-norm is cross (Proposition II.11.7), by Proposition II.11.5 $\overline{(\mathfrak{B}_o \mathbin{\textcircled{u}} \mathfrak{C}_o) \mathbin{\textcircled{u}} \mathfrak{D}_o}^{\min}$ contains isomorphic copies of the completion of its marginals C^*-algebras i.e.

$$
\mathfrak{B} \mathbin{\textcircled{u}} \mathfrak{C}, \mathfrak{D} \hookrightarrow \overline{(\mathfrak{B}_o \mathbin{\textcircled{u}} \mathfrak{C}_o) \mathbin{\textcircled{u}} \mathfrak{D}_o}^{\min}
$$

Analogously, $\mathfrak{B}, \mathfrak{C} \mathbin{\textcircled{u}} \mathfrak{D} \hookrightarrow \overline{\mathfrak{B}_o \mathbin{\textcircled{u}} (\mathfrak{C}_o \mathbin{\textcircled{u}} \mathfrak{D}_o)}^{\min}$. In conclusion,

$$
(\mathfrak{B} \mathbin{\textcircled{u}} \mathfrak{C}) \mathbin{\textcircled{u}} \mathfrak{D} \cong \mathfrak{B} \mathbin{\textcircled{u}} (\mathfrak{C} \mathbin{\textcircled{u}} \mathfrak{D})\,. \qquad \square
$$

In view of Lemma III.3.1, we are allowed to write $\mathfrak{B} \mathbin{\textcircled{u}} \mathfrak{C} \mathbin{\textcircled{u}} \mathfrak{D}$ with no issues of ambiguity. In particular, we will be able to construct the infinite twisted C^*-tensor product out of a single C^*-system $(\mathfrak{B}, G, \beta)$, with no concern for where to put brackets. We start by recursively define

$$
\begin{aligned}
&\mathfrak{A}_1 := \mathfrak{B}\,, \quad \mathfrak{A}_{n+1} := \mathfrak{A}_n \mathbin{\textcircled{u}} \mathfrak{B}\,, \\
&\iota_n : \mathfrak{A}_n \hookrightarrow \mathfrak{A}_{n+1}\,, \quad n \in \mathbb{N}
\end{aligned}
$$

where ι_n is a well defined *-monomorphism thanks to Proposition II.11.5. After defining the *connecting maps* $\phi_{nm} := \iota_{m-1} \circ \cdots \circ \iota_{n+1} \circ \iota_n$ $(n < m)$ and $\phi_{nn} := I_{\mathfrak{A}_n}$ for each $n \in \mathbb{N}$, it is immediate to verify that $(\mathfrak{A}_n, \phi_{nm})_{n \le m}$ is a direct system of C^*-algebras over $\mathbb{N}$, thus yielding the *direct limit* unital *-algebra

$$
\mathfrak{A}_\infty := \varinjlim_{n} \mathfrak{A}_n\,.
$$

It is well known that $\mathfrak{A}_\infty$ comes with a family of *canonical maps* $\jmath_n \colon \mathfrak{A}_n \to \mathfrak{A}_\infty$ $(n \in \mathbb{N})$ s.t. $\jmath_m \circ \phi_{nm} = \jmath_n$ for every $n \le m$, and that $\mathfrak{A}_\infty$ satisfies a universal property: if A is a unital *-algebra admitting a family of unital *-homomorphisms $\eta_n \colon \mathfrak{A}_n \to A$ satisfying $\eta_m \circ \phi_{nm} = \eta_n$ $(m, n \in \mathbb{N},\ n \le m)$, then there exists a unique unital *-homomorphism $\Gamma \colon \mathfrak{A}_\infty \to A$ s.t. $\eta_n = \Gamma \circ \jmath_n$ for every $n \in \mathbb{N}$.

Now, set

$$
\alpha_g^{(1)} := \beta_g \in \mathsf{Aut}(\mathfrak{A}_1),\ \ \alpha_g^{(n+1)} := \alpha_g^{(n)} \mathbin{\textcircled{u}} \beta_g \in \mathsf{Aut}(\mathfrak{A}_{n+1}), \quad g \in G,\ n \in \mathbb{N}
$$

Then, $\alpha^{(n)} \colon g \mapsto \alpha_g^{(n)}$ defines a pointwise norm-continuous action of G on $\mathfrak{A}_n$ for each $n \in \mathbb{N}$. Furthermore, the automorphisms $\left(\alpha_g^{(n)}\right)_{\substack{n\in\mathbb{N}\\ g\in G}}$ evidently satisfy the relations

$$
\alpha_g^{(m)} \circ \phi_{nm} = \phi_{nm} \circ \alpha_g^{(n)}, \quad n \le m,\ g \in G\,. \tag{III.5}
$$

If we fix $g \in G$ and take $A := \mathfrak{A}_\infty$, $\eta_{n,g} := \jmath_n \circ \alpha_g^{(n)}$ $(n \in \mathbb{N})$, by using Equation III.5 we immediately see that

$$
\eta_{m,g} \circ \phi_{nm} = \jmath_m \circ \alpha_g^{(m)} \circ \phi_{nm} = \jmath_m \circ \phi_{nm} \circ \alpha_g^{(n)} = \jmath_n \circ \alpha_g^{(n)} = \eta_{n,g}
$$

hence by the universal property of $\mathfrak{A}_\infty$, there exists a unique unital *-endomorphism $\Gamma := \alpha_g^\infty \in \mathsf{End}(\mathfrak{A}_\infty)$. Not only that, for every $g \in G$, $\alpha_g^{(\infty)} \in \mathsf{Aut}(\mathfrak{A}_\infty)$ (see Lemma L.1.3 at p. 300 in [108]), thus providing an action $G \overset{\alpha^{(\infty)}}{\curvearrowright} \mathfrak{A}_\infty$. The non-negative quantity

$$\|\jmath_n(x_n)\|_{\mathfrak{A}_\infty} := \|x_n\|_{\mathfrak{A}_n}, \quad x_n \in \mathfrak{A}_n,\ n \in \mathbb{N}$$

defines a C^*-norm on $\mathfrak{A}_\infty$, inducing the final topology associated to the canonical maps $\jmath_n$'s. The completion of $\mathfrak{A}_\infty$ w.r.t. $\|\cdot\|_{\mathfrak{A}_\infty}$ is the *C^*-inductive limit* $\mathfrak{A}$ of the above direct system $(\mathfrak{A}_n, \phi_{nm})_{n\le m}$. It is endowed with a family of canonical isometric maps, again denoted by $\jmath_n \colon \mathfrak{A}_n \hookrightarrow \mathfrak{A}$ $(n \in \mathbb{N})$, and its universal property follows *verbatim* the one of $\mathfrak{A}_\infty$, as long as we require A to be a unital C^*-algebra. Moreover, the following relations hold (see Proposition L.2.2 at pp. 303-304 in [108]):

$$\begin{aligned}
\mathfrak{A}_{\mathrm{sa}} &= \overline{\bigcup_{n\in\mathbb{N}} \jmath_n\left(\mathfrak{A}_{n,\mathrm{sa}}\right)} & \mathcal{U}(\mathfrak{A}) &= \overline{\bigcup_{n\in\mathbb{N}} \jmath_n\left(\mathcal{U}(\mathfrak{A}_n)\right)} \\
\mathfrak{A}_{+} &= \overline{\bigcup_{n\in\mathbb{N}} \jmath_n\left(\mathfrak{A}_{n,+}\right)} & \mathcal{L}(\mathfrak{A}) &= \overline{\bigcup_{n\in\mathbb{N}} \jmath_n\left(\mathcal{L}(\mathfrak{A}_n)\right)} \\
\mathfrak{A}_{+} \cap B_{\mathfrak{A}} &= \overline{\bigcup_{n\in\mathbb{N}} \jmath_n\left(\mathfrak{A}_{n,+} \cap B_{\mathfrak{A}_n}\right)} & GL(\mathfrak{A}) &\subsetneq \overline{\bigcup_{n\in\mathbb{N}} \jmath_n(GL(\mathfrak{A}_n))}
\end{aligned}$$

where $B_{\mathfrak{A}}$ (respectively, $B_{\mathfrak{A}_n}$) is the closed unit ball of $\mathfrak{A}$ ($\mathfrak{A}_n$) and $\mathcal{L}(\mathfrak{A})$ (respectively, $\mathcal{L}(\mathfrak{A}_n)$) is the lattice of the orthogonal projections in $\mathfrak{A}$ ($\mathfrak{A}_n$). Recall also that $GL(\mathfrak{A})$ and $GL(\mathfrak{A}_n)$ are open.
Lastly, for every $g \in G$, $\alpha_g^{(\infty)} \in \mathsf{Aut}(\mathfrak{A}_\infty)$ is $\|\cdot\|_{\mathfrak{A}_\infty}$-isometric, thus extendable to a C^*-automorphism $\alpha_g \in \mathsf{Aut}(\mathfrak{A})$. It is immediate to check that the map $g \mapsto \alpha_g$ is pointwise norm-continuous, thus yielding an action $G \overset{\alpha}{\curvearrowright} \mathfrak{A}$. We collect the previous results in the following

Theorem III.3.2
The inductive sequence of C^*-algebras $(\mathfrak{A}_n)_n$, together with the compatible sequence of actions $\left(G \overset{\alpha^{(n)}}{\curvearrowright} \mathfrak{A}_n\right)_n$, determines a C^*-system $(\mathfrak{A}, G, \alpha)$, referred to as the (minimal) *C^*-inductive limit* of a (countably) infinite number of copies of a single C^*-system $(\mathfrak{B}, G, \beta)$.

We will also refer to $\mathfrak{A}$ in the previous theorem as the *twisted chain of* $\mathfrak{B}$, of which $\mathfrak{A}_\infty$ is the (dense, involutive) algebra of *localized* elements. It admits a $\widehat{G}$-graded structure, where for every $\sigma \in \widehat{G}$, the σ-spectral subspace of $\mathfrak{A}$ is

$$\mathfrak{A}_\sigma = \left[\jmath_n(\mathfrak{B}_{(\sigma_1,\ldots,\sigma_n)}) \colon \sum_{i=1}^{n} \sigma_i = \sigma, n \in \mathbb{N}\right]^{[2]}, \quad \mathfrak{B}_{(\sigma_1,\ldots,\sigma_n)} := \mathfrak{B}_{\sigma_1} \odot \cdots \odot \mathfrak{B}_{\sigma_n} \subseteq \mathfrak{A}_n\,.$$

We conclude the section with a clarification. For any $n \in \mathbb{N}$, there are exactly $C_{n-1} := \dfrac{1}{n}\dbinom{2(n-1)}{n-1}$ ways of inserting brackets in a chain of n copies of $\mathfrak{B}$ in order to associate the $n-1$ products ⓦ among them.[3] For example, for $n = 4$, there are $C_3 = 5$ distinct ways to parenthesize a chain of 4 copies of $\mathfrak{B}$:

[2]For a subset S, $[S] := \overline{\mathrm{span}_{\mathbb{C}}\, S}$.
[3]C_m is called the m^{th} *Catalan number*.

$$(\mathfrak{B} \textcircled{u} \mathfrak{B}) \textcircled{u} (\mathfrak{B} \textcircled{u} \mathfrak{B}),\ ((\mathfrak{B} \textcircled{u} \mathfrak{B}) \textcircled{u} \mathfrak{B}) \textcircled{u} \mathfrak{B} = \mathfrak{A}_4,$$
$$(\mathfrak{B} \textcircled{u} (\mathfrak{B} \textcircled{u} \mathfrak{B})) \textcircled{u} \mathfrak{B},\ \mathfrak{B} \textcircled{u} ((\mathfrak{B} \textcircled{u} \mathfrak{B}) \textcircled{u} \mathfrak{B}),\ \mathfrak{B} \textcircled{u} (\mathfrak{B} \textcircled{u} (\mathfrak{B} \textcircled{u} \mathfrak{B}))\,.$$

In view of Lemma III.3.1, all the five C^*-algebras above are isomorphic. More in general, for every $n \in \mathbb{N}$, all the C_n possible parenthesizations are isomorphic C^*-algebras. The previous considerations allows us to simply write $\mathfrak{A}_n = \underbrace{\mathfrak{B} \textcircled{u} \cdots \textcircled{u} \mathfrak{B}}_{n \text{ copies}}$.
Analogously, $\mathfrak{A}_\infty = \varinjlim_n \underbrace{\mathfrak{B} \textcircled{u} \cdots \textcircled{u} \mathfrak{B}}_{n \text{ copies}}$. It is then meaningful to write $\textcircled{u}_{n\in\mathbb{N}} \mathfrak{B} := \overline{\mathfrak{A}_\infty} = \mathfrak{A}$, exactly as for the usual tensor product (see p. 18-28 in [43]).

III.4. The flip maps Φ and Φ_u

To avoid ambiguities, we will temporarily assume $\mathsf{S}(\widehat{G}) \cap \mathsf{A}(\widehat{G}) = (\mathbb{1}_{\widehat{G}\times\widehat{G}})$. This requirement is equivalent to ask that $\widehat{G}$ is 2-*divisible* (i.e. $\widehat{G} = \widehat{G}^2$), a condition which is satisfied in a variety of cases, such as $\widehat{G}$ finite with odd order, the discrete real line $\mathbb{R}_\mathrm{d}$, the discrete circle $\mathbb{R}_\mathrm{d}/\mathbb{Z}$, and the additive rationals $\mathbb{Q}$. (On the other hand, $\widehat{G}$ is evidently not 2-divisible when having even order, or for $\widehat{G} = \mathbb{Z}^n$; we will see what happens in these cases later on.)
Given non-trivial $u \in \mathsf{B}(\widehat{G})$, we define the (respectively, *untwisted* and *twisted*) *flip* maps as the $\mathbb{C}$-linear extensions of

$$\begin{aligned}\Phi\colon \mathfrak{B}_o \textcircled{u} \mathfrak{B}_o &\to \mathfrak{B}_o \textcircled{u} \mathfrak{B}_o \\ a \textcircled{u} b &\mapsto b \textcircled{u} a\end{aligned} \qquad \begin{aligned}\Phi_u\colon \mathfrak{B}_o \textcircled{u} \mathfrak{B}_o &\to \mathfrak{B}_o \textcircled{u} \mathfrak{B}_o \\ A \textcircled{u} B &\mapsto u(A,B)B \textcircled{u} A \quad (A, B \text{ homogeneous}).\end{aligned}$$

Our purpose is to investigate the properties of Φ and Φ_u, according to the ones imposed on the algebra $\mathfrak{B}$ and the bicharacter u. Precisely, we aim to show the following scheme:

*-preservation		
Flip	$u \in \mathsf{S}(\widehat{G})$	$u \in \mathsf{A}(\widehat{G})$
Φ	✓	✗
Φ_u	✗	✓

product-preservation		
Flip	$u \in \mathsf{S}(\widehat{G})$	$u \in \mathsf{A}(\widehat{G})$
Φ	✗	✗
Φ_u	✗	✓

product-reversal, $\mathfrak{B}$ abelian		
Flip	$u \in \mathsf{S}(\widehat{G})$	$u \in \mathsf{A}(\widehat{G})$
Φ	✓	✗
Φ_u	✗	✗

To accomplish that, consider any function $f\colon \widehat{G} \times \widehat{G} \to \mathbb{T}$ and the self-map

$$\begin{aligned}\Phi_f\colon \mathfrak{B}_o \textcircled{u} \mathfrak{B}_o &\to \mathfrak{B}_o \textcircled{u} \mathfrak{B}_o \\ a \odot b &\mapsto f_{\partial a, \partial b}\, b \odot a \quad (a, b \text{ homogeneous})\end{aligned}$$

Then, one straightforwardly checks that Φ_f always intertwines the direct product action of $G \times G$ on $\mathfrak{B}_o \textcircled{u} \mathfrak{B}_o$, i.e.

$$\Phi_f \circ (\beta_g \textcircled{u} \beta_{g'}) = (\beta_{g'} \textcircled{u} \beta_g) \circ \Phi_f, \quad g, g' \in G \tag{III.6}$$

and that it is

(i) *-preserving if $f_{\sigma,\tau} f_{\sigma^{-1},\tau^{-1}} = u(\sigma,\tau)\overline{u(\tau,\sigma)}$ $(\sigma, \tau \in \widehat{G})$

(ii) product-preserving if $f_{\sigma,\tau} f_{\xi,\eta} \overline{f_{\sigma\xi,\tau\eta}} = u(\eta,\sigma)\overline{u(\xi,\tau)}$ $(\sigma, \tau, \xi, \eta \in \widehat{G})$

(iii) product-reversing (when $\mathfrak{B}$ is abelian) if $f_{\sigma,\tau} f_{\xi,\eta} \overline{f_{\sigma\xi,\tau\eta}} = u(\tau,\xi)\overline{u(\xi,\tau)}$ $(\sigma, \tau, \xi, \eta \in \widehat{G})$

Therefore, Φ corresponds to the case $f = \mathbb{1}_{\widehat{G}\times\widehat{G}}$ (i.e. $\Phi = \Phi_1$), whereas Φ_u to the case $f = u$. Notice that the sufficient conditions here listed become also *necessary* if for every pair of characters $\sigma, \tau \in \widehat{G}$, $\mathfrak{B}_\sigma\mathfrak{B}_\tau \neq \{0\}$ (a condition which is, in general, stronger than the G-action having full spectrum on $\mathfrak{B}$).
When $u \in \mathsf{S}(\widehat{G})$, $\Phi = \Phi_1$ clearly satisfies condition (i) (for any C^*-algebra $\mathfrak{B}$) and condition (iii) (for $\mathfrak{B}$ abelian), while (ii) fails to hold since u is non-trivial. On the other hand, Φ_u does not comply with any of the three conditions. On the contrary, when $u \in \mathsf{A}(\widehat{G})$, Φ_u satisfies conditions (i) and (ii), whereas $\Phi = \Phi_1$ is compatible with none of (i), (ii), (iii).
We collect these results in the following

Lemma III.4.1
If $u \in \mathsf{A}(\widehat{G})$, Φ_u isometrically extends to an involutive *-automorphism (again denoted by Φ_u) of $\mathfrak{B} \textcircled{u} \mathfrak{B}$, intertwining the direct product action of $G \times G$ on $\mathfrak{B} \textcircled{u} \mathfrak{B}$. In particular, every Φ_u-invariant state $\varphi \in \mathcal{S}(\mathfrak{B} \textcircled{u} \mathfrak{B})$ induces a covariant *unitary* GNS representation $(\mathcal{H}_\varphi, \pi_\varphi, U_\varphi, \xi_\varphi)$ of $\mathfrak{B} \textcircled{u} \mathfrak{B}$, where $U_\varphi^2 = I$ on $\mathcal{H}_\varphi$.
If $u \in \mathsf{S}(\widehat{G})$ and $\mathfrak{B}$ is abelian, Φ isometrically extends to an involutive *-anti-automorphism (again denoted by Φ) of $\mathfrak{B} \textcircled{u} \mathfrak{B}$, intertwining the direct product action of $G \times G$ on $\mathfrak{B} \textcircled{u} \mathfrak{B}$. In particular, every Φ-invariant state $\varphi \in \mathcal{S}(\mathfrak{B} \textcircled{u} \mathfrak{B})$ induces a covariant *anti-unitary* GNS representation $(\mathcal{H}_\varphi, \pi_\varphi, \widetilde{U}_\varphi, \xi_\varphi)$ of $\mathfrak{B} \textcircled{u} \mathfrak{B}$, where $\widetilde{U}_\varphi^2 = I$ on $\mathcal{H}_\varphi$.

Proof.
By the previous discussion, if $u \in \mathsf{A}(\widehat{G})$, Φ_u is an (involutive) *-automorphism of $\mathfrak{B}_o \textcircled{u} \mathfrak{B}_o$, so let us check that it is min-isometric. Firstly, observe that since $u \in \mathsf{A}(\widehat{G})$, for every pair of characters $\sigma, s \in \widehat{G}$ we have $s(g_\sigma) = s({}_\sigma g) = \overline{\sigma(g_s)} = \overline{\sigma({}_s g)}$, and in particular $g_\sigma = {}_\sigma g \in G$ for every $\sigma \in \widehat{G}$. If $\omega, \varphi \in \mathcal{S}_G(\mathfrak{B})$ and $\Sigma_{\varphi,\omega} \colon \mathcal{H}_\varphi \otimes \mathcal{H}_\omega \to \mathcal{H}_\omega \otimes \mathcal{H}_\varphi$ is the swapping unitary operator on the Hilbert tensor products of the associated GNS spaces, then for each homogeneous $a, b \in \mathfrak{B}$,

$$
\begin{aligned}
[(\pi_{\omega U_\omega} \textcircled{u} \pi_\varphi) \circ \Phi_u](a \textcircled{u} b) &= u(a,b)\pi_\omega(b)U_\omega(g_{\partial a}) \otimes \pi_\varphi(a) = u(a,b)\Sigma_{\varphi,\omega}\pi_\varphi(a) \otimes \pi_\omega(b)U_\omega(g_{\partial a})\Sigma^*_{\varphi,\omega} = \\
&= u(a,b)\Sigma_{\varphi,\omega}\pi_\varphi(a) \otimes U_\omega(g_{\partial a})\pi_\omega(\beta^{-1}_{g_{\partial a}}(b))\Sigma^*_{\varphi,\omega} = \Sigma_{\varphi,\omega}\pi_\varphi(a) \otimes U_\omega({}_{\partial a}g)\pi_\omega(b)\Sigma^*_{\varphi,\omega} = \\
&= \Sigma_{\varphi,\omega}(\pi_\varphi \textcircled{u}\, {}_{U_\omega}\pi_\omega)(a \textcircled{u} b)\Sigma^*_{\varphi,\omega}\,.
\end{aligned}
$$

The equality above easily extends to every $x \in \mathfrak{B}_o \textcircled{u} \mathfrak{B}_o$, hence giving

$$
\|\Phi_u(x)\|_{\min} = \sup_{\omega,\varphi \in \mathcal{S}_G(\mathfrak{A})} \|(\pi_{\omega\times\varphi} \circ \Phi_u)(x)\|_{\mathcal{B}(\mathcal{H}_\omega \otimes \mathcal{H}_\varphi)} = \sup_{\varphi,\omega \in \mathcal{S}_G(\mathfrak{A})} \|\pi_{\varphi\times\omega}(x)\|_{\mathcal{B}(\mathcal{H}_\varphi \otimes \mathcal{H}_\omega)} = \|x\|_{\min}\,.
$$

In particular, the isometric extension of Φ_u (again denoted by Φ_u) is an involutive *-automorphism of $\mathfrak{B} \textcircled{u} \mathfrak{B}$. Lastly, if $\varphi \in \mathcal{S}(\mathfrak{B} \textcircled{u} \mathfrak{B})$ is Φ_u-invariant, the densely defined operator

$$
U_\varphi(\pi_\varphi(x)\xi_\varphi) := (\pi_\varphi \circ \Phi_u)(x)\xi_\varphi, \quad x \in \mathfrak{B} \textcircled{u} \mathfrak{B}
$$

extends to a selfadjoint (equivalently, involutive) unitary operator $U_\varphi \in \mathcal{U}(\mathcal{H}_\varphi)$ s.t. $U_\varphi\xi_\varphi = \xi_\varphi$ and $U_\varphi\pi_\varphi(x)U_\varphi = (\pi_\varphi \circ \Phi_u)(x)$, $x \in \mathfrak{B} \textcircled{u} \mathfrak{B}$ (see [59], Lemma 2.1, p. 11).
If $u \in \mathsf{S}(\widehat{G})$ and $\mathfrak{B}$ is abelian, by the previous discussion Φ is an (involutive) *-anti-automorphism of $\mathfrak{B}_o \textcircled{u} \mathfrak{B}_o$. Furthermore, Φ induces a compatible C^*-norm $\|\cdot\|_\Phi := \|\Phi(\cdot)\|_{\min}$ on $\mathfrak{B}_o \textcircled{u} \mathfrak{B}_o$ since

$$
\|(\beta_g \textcircled{u} \beta_{g'})(x)\|_\Phi = \|\Phi \circ (\beta_g \textcircled{u} \beta_{g'})(x)\|_{\min} = \|(\beta_{g'} \textcircled{u} \beta_g) \circ \Phi(x)\|_{\min} = \|\Phi(x)\|_{\min} = \|x\|_\Phi
$$

for every $g, g' \in \mathfrak{B}_o \textcircled{u} \mathfrak{B}_o$. Thus, by Theorem II.17.4, $\|\cdot\|_\Phi = \|\cdot\|_{\min}$ and the isometric extension of Φ (again denoted by Φ) is an involutive *-anti-automorphism of $\mathfrak{B} \textcircled{u} \mathfrak{B}$. Lastly, if $\varphi \in \mathcal{S}(\mathfrak{B} \textcircled{u} \mathfrak{B})$ is invariant under Φ, the densely defined operator

$$
\widetilde{U}_\varphi(\pi_\varphi(x)\xi_\varphi) := (\pi_\varphi \circ \Phi)(x^*)\xi_\varphi, \quad x \in \mathfrak{B} \textcircled{u} \mathfrak{B}
$$

extends to a $\mathbb{C}$-antilinear operator $\widetilde{U}_\varphi \colon \mathcal{H}_\varphi \to \mathcal{H}_\varphi$ s.t. $\langle \widetilde{U}_\varphi \xi, \widetilde{U}_\varphi \eta\rangle_{\mathcal{H}_\varphi} = \overline{\langle \xi, \eta\rangle}_{\mathcal{H}_\varphi}$ and $\langle \widetilde{U}_\varphi \xi, \eta\rangle = \overline{\langle \xi, \widetilde{U}_\varphi \eta\rangle}_{\mathcal{H}_\varphi}$ for every $\eta, \xi \in \mathcal{H}_\varphi$, i.e. a selfadjoint (equivalently, involutive) anti-unitary operator $\widetilde{U}_\varphi$. Moreover, $\widetilde{U}_\varphi \xi_\varphi = \xi_\varphi$ and $\widetilde{U}_\varphi \pi_\varphi(x^*)\widetilde{U}_\varphi = (\pi_\varphi \circ \Phi)(x)$, $x \in \mathfrak{B} \textcircled{u} \mathfrak{B}$. $\square$

Remark III.4.2
The previous lemma provides:

- a C^*-system $(\mathfrak{B} \textcircled{u} \mathfrak{B}, \mathbb{Z}_2, \Phi_u)$ for $u \in \mathsf{A}(\widehat{G})$, where each *even* state $\varphi \in \mathcal{S}_{\mathbb{Z}_2}(\mathfrak{B} \textcircled{u} \mathfrak{B})$ induces a $\mathbb{Z}_2$-grading on its GNS Hilbert space $\mathcal{H}_\varphi = \mathcal{H}_{\varphi,+} \oplus \mathcal{H}_{\varphi,-} := \ker(U_\varphi - I) \oplus \ker(U_\varphi + I)$

- a pair $(\mathfrak{B} \textcircled{u} \mathfrak{B}, \Phi)$ for $u \in \mathsf{S}(\widehat{G})$ and $\mathfrak{B}$ abelian, where each Φ-invariant state $\varphi \in \mathcal{S}_\Phi(\mathfrak{B} \textcircled{u} \mathfrak{B})$ induces a decomposition of its GNS Hilbert space $\mathcal{H}_\varphi = \widetilde{\mathcal{K}}_{\varphi,+} \oplus \widetilde{\mathcal{K}}_{\varphi,-} = \widetilde{\mathcal{K}}_{\varphi,+} \oplus i\,\widetilde{\mathcal{K}}_{\varphi,+}$, where $\widetilde{\mathcal{K}}_{\varphi,\pm} := \{\xi \in \mathcal{H}_\varphi \colon \widetilde{U}_\varphi \xi = \pm\xi\}$ are merely $\mathbb{R}$-linear closed subspaces of $\mathcal{H}_\varphi$

For a general (non-trivial) abelian discrete group $\widehat{G}$, if $u \in \mathsf{S}(\widehat{G}) \cap \mathsf{A}(\widehat{G})$ is non-degenerate, then clearly $\operatorname{im}(u) = \{\pm 1\} \cong \mathbb{Z}_2$. (We remark that the viceversa is not true: $u(\underline{\sigma}, \underline{\tau}) = (-1)^{\underline{\sigma}^t \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \underline{\tau}}$ for $\underline{\sigma}, \underline{\tau} \in \mathbb{Z}_2^2$ is a non-degenerate bicharacter s.t. $\operatorname{im}(u) = \{\pm 1\}$, but $u \notin \mathsf{S}(\mathbb{Z}_2^2) = \mathsf{A}(\mathbb{Z}_2^2)$.) Therefore, requiring u to belong to $\mathsf{S}(\widehat{G}) \setminus \mathsf{A}(\widehat{G})$ is equivalent to have $u \in \mathsf{S}(\widehat{G})$ s.t. $\{\pm 1\} \subsetneq \operatorname{im}(u)$. In such a case, the first column in the three tables above can be filled in likewise. Analogously, $u \in \mathsf{A}(\widehat{G}) \setminus \mathsf{S}(\widehat{G})$ if and only if $u \in \mathsf{A}(\widehat{G})$ and $\{\pm 1\} \subsetneq \operatorname{im}(u)$, in which case the second column is completed in the same manner.
In view of Lemma III.4.1, one might be led to think that, up to substituting unitary operators with anti-unitary ones, all the ergodic theory of symmetric states on a twisted chain of an abelian C^*-algebra $\mathfrak{B}$ could be performed also in the case when $u \in \mathsf{S}(\widehat{G}) \setminus \mathsf{A}(\widehat{G})$. Unfortunately, this hope is promptly dampened by the fact that, when adding a third copy of $\mathfrak{B}$ to the tensor product, the mapping $\Phi \textcircled{u} I_{\mathfrak{B}_o}$ is in general just a unital, *-preserving, involutive self-map on $(\mathfrak{B}_o \textcircled{u} \mathfrak{B}_o) \textcircled{u} \mathfrak{B}_o$, with no further satisfying properties, not even positivity. To motivate this statement, we observe that the following requirements are equivalent:

- $\Phi \textcircled{u} I_{\mathfrak{B}_o}$ is min-contractive on $\mathfrak{B}_o \textcircled{u} \mathfrak{B}_o \textcircled{u} \mathfrak{B}_o$

- $\Phi \textcircled{u} I_{\mathfrak{B}_o}$ is min-isometric on $\mathfrak{B}_o \textcircled{u} \mathfrak{B}_o \textcircled{u} \mathfrak{B}_o$

- $\Phi \textcircled{u} I_{\mathfrak{B}_o}$ extends to an *order* automorphism of $\mathfrak{B} \textcircled{u} \mathfrak{B} \textcircled{u} \mathfrak{B}$

- $\Phi \textcircled{u} I_{\mathfrak{B}_o}$ extends to a *Jordan* automorphism of $\mathfrak{B} \textcircled{u} \mathfrak{B} \textcircled{u} \mathfrak{B}$

For instance, if $\mathfrak{B} := \mathcal{C}(\mathbb{T})$ and $u_\alpha(m,n) := e^{i2\pi\alpha mn}$ (α irrational, $m, n \in \mathbb{Z}$), then $u_\alpha \in \mathsf{S}(\widehat{G}) \setminus \mathsf{A}(\widehat{G})$ and Φ results to be an involutive *-anti-automorphism of the rotation algebra $\mathbb{A}_\alpha = \mathcal{C}(\mathbb{T}) \textcircled{u} \mathcal{C}(\mathbb{T})$. However, $\Phi \textcircled{u} I_{\mathcal{C}(\mathbb{T})_o}$ on $(\mathcal{C}(\mathbb{T})_o \textcircled{u} \mathcal{C}(\mathbb{T})_o) \textcircled{u} \mathcal{C}(\mathbb{T})_o$ cannot have a positive extension to the minimal completion. By contradiction, if it did, by the previous discussion it would be a Jordan automorphism of $\mathcal{C}(\mathbb{T}) \textcircled{u} \mathcal{C}(\mathbb{T}) \textcircled{u} \mathcal{C}(\mathbb{T})$. We straightforwardly see that this is not the case, by exhibiting $x, y \in \mathcal{C}(\mathbb{T})_o \textcircled{u} \mathcal{C}(\mathbb{T})_o \textcircled{u} \mathcal{C}(\mathbb{T})_o$ s.t. $\Phi \textcircled{u} I_{\mathcal{C}(\mathbb{T})_o}(\{x,y\}) \neq \{(\Phi \textcircled{u} I_{\mathcal{C}(\mathbb{T})_o})(x), (\Phi \textcircled{u} I_{\mathcal{C}(\mathbb{T})_o})(y)\}$. Let (U, V, W) the ordered triplet of generators of $\mathcal{C}(\mathbb{T})_o \textcircled{u} \mathcal{C}(\mathbb{T})_o \textcircled{u} \mathcal{C}(\mathbb{T})_o$ and $x := UW$, $y := V^2 W$. Then,

$$\Phi \textcircled{u} I_{\mathcal{C}(\mathbb{T})_o}(\{x,y\}) = \Phi \textcircled{u} I_{\mathcal{C}(\mathbb{T})_o}((e^{-i4\pi\alpha} + e^{-i6\pi\alpha}) U V^2 W^2) = e^{-i2\pi\alpha}(e^{-i2\pi\alpha} + e^{-i4\pi\alpha}) U^2 V W^2$$

$$\{(\Phi \textcircled{u} I_{\mathcal{C}(\mathbb{T})_o})(x), (\Phi \textcircled{u} I_{\mathcal{C}(\mathbb{T})_o})(y)\} = (VW)(U^2 W) + (U^2 W)(VW) = e^{-i2\pi\alpha}(e^{-i6\pi\alpha} + 1) U^2 V W^2 \,.$$

This is the reason why, from now on, we will restrict our analysis to $u \in \mathsf{A}(\widehat{G})$, in which case all the (embedded) flips Φ_u will act as *-automorphisms on a twisted chain of a C^*-algebra $\mathfrak{B}$, then inducing a well-defined action of the finitary symmetric group $\mathfrak{S}$.

III.5. Non-degenerate skew-symmetric bicharacters

In this section, all the abelian groups $\mathcal{G}$ we deal with are *finite* (hence, direct products of cyclic subgroups of prime-power order). For $u \in \mathsf{B}(\mathcal{G})$ and any subgroup $H \leq \mathcal{G}$, we write (by a slight abuse of notation) $u|_H$ for the restriction of u to the direct product $H \times H$. Clearly, $u|_H \in \mathsf{B}(H)$. Moreover, if $u_i \in \mathsf{B}(\mathcal{G}_i)$ $(i = 1, 2)$, we define $u_1 \otimes u_2 \in \mathsf{B}(\mathcal{G}_1 \times \mathcal{G}_2)$ as

$$(u_1 \otimes u_2)((g_1, g_2), (g_1', g_2')) := u_1(g_1, g_1')u_2(g_2, g_2'), \quad g_i, g_i' \in \mathcal{G}_i \tag{III.7}$$

We say that $(\mathcal{G}_1, u_1)$ and $(\mathcal{G}_2, u_2)$ are *equivalent* (and write $u_1 \sim u_2$) if there exists a group isomorphism $T\colon \mathcal{G}_1 \xrightarrow{\sim} \mathcal{G}_2$ s.t. $u_1(g, h) = u_2(T(g), T(h))$, $g, h \in \mathcal{G}_1$. A skew-symmetric bicharacter $u \in \mathsf{A}(\mathcal{G})$ is *reducible* if there exist two pairs $(\mathcal{G}_i, u_i)$, $i = 1, 2$, s.t. $\begin{cases} \mathcal{G}_1, \mathcal{G}_2 \ncong (0) \\ \mathcal{G} \cong \mathcal{G}_1 \times \mathcal{G}_2 \\ u \sim u_1 \otimes u_2 \end{cases}$.
Lastly, a non-degenerate bicharacter $u \in \mathsf{A}(\mathcal{G})$ is *elementary* if $(u, \mathcal{G})$ is equivalent to one of the following three pairs:

(1) *Fermi* bicharacter of $\mathbb{Z}_2$

$(\mathbb{Z}_2, u_{\mathtt{F}})$, $u_{\mathtt{F}}(x, y) := (-1)^{xy}$, where $x, y \in \mathbb{Z}_2 = \{0, 1\}$.
In this case, $\Delta_+ = (0)$.

(2) *Non-symplectic* bicharacters of the 2-primary groups $\mathbb{Z}_{2^n}^2$

$(\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n}, u_{2^n})$, $u_{2^n}(\mathbf{x}, \mathbf{y}) := \xi^{\mathbf{x}^t \left[\begin{smallmatrix} 2^{n-1} & 1 \\ -1 & 0 \end{smallmatrix}\right] \mathbf{y}}$, where

- $n \geq 1$ and $\mathbf{x}, \mathbf{y} \in \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n}$
- $\xi \in \mathbb{T}$ is a *primitive* 2^n-th root of unity (i.e. $\xi^{(2^n)} = 1$ and $n = \min\{k \in \mathbb{N}\colon \xi^{(2^k)} = 1\}$)

In this case, $\Delta_+ = \langle 2 \rangle \times \mathbb{Z}_{2^n} \cong \mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_{2^n}$.

(3) *Symplectic* bicharacters of the p-primary groups $\mathbb{Z}_{p^n}^2$

$(\mathbb{Z}_{p^n} \times \mathbb{Z}_{p^n}, w_\zeta)$, $w_{p^n}(\mathbf{x}, \mathbf{y}) := \zeta^{\mathbf{x}^t \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right] \mathbf{y}}$ where

- $n \geq 1$, $p \geq 2$ is prime and $\mathbf{x}, \mathbf{y} \in \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^n}$
- $\zeta \in \mathbb{T}$ is a *primitive* p^n-th root of unity (i.e. $\zeta^{(p^n)} = 1$ and $n = \min\{k \in \mathbb{N}\colon \zeta^{(p^k)} = 1\}$)

In this case, $\Delta_+ = \mathbb{Z}_{p^n}^2$.

Recall that a p^n-th root of unity $\zeta \in \mathbb{T}$ is primitive if and only if $1, \zeta, \cdots, \zeta^{p^n - 1} \in \mathbb{T}$ are all distinct. Equivalently, $\zeta = e^{i\frac{2\pi}{p^n}L}$ for some $L = 1, \ldots, p^n$ s.t. $\gcd(p^n, L) = 1$ i.e. $p \nmid L$. In other words, $L \notin \{kp\colon k = 1, \ldots, p^{n-1}\}$, so that there are exactly $\varphi(p^n) = p^n - p^{n-1} = p^n\left(1 - \frac{1}{p}\right)$ totatives of p^n, each corresponding to a distinct p^n-th primitive root of unity (φ is the Euler's totient function). Let $U(\mathbb{Z}_{p^n})$ be the multiplicative group of totatives of p^n (i.e. units of the commutative, unital ring $\mathbb{Z}_{p^n}$). Then, we can re-write

$$u_{2^n}(\mathbf{x}, \mathbf{y}) = e^{i\frac{2\pi}{2^n}\mathbf{x}^t \left[\begin{smallmatrix} 2^{n-1} & K \\ -K & 0 \end{smallmatrix}\right] \mathbf{y}} = e^{i\frac{2\pi}{2^n} b_K(\mathbf{x}, \mathbf{y})}, \quad K \in U(\mathbb{Z}_{2^n})$$

$$w_{p^n}(\mathbf{x}, \mathbf{y}) = e^{i\frac{2\pi}{p^n}\mathbf{x}^t \left[\begin{smallmatrix} 0 & L \\ -L & 0 \end{smallmatrix}\right] \mathbf{y}} = e^{i\frac{2\pi}{2^n} c_L(\mathbf{x}, \mathbf{y})}, \quad L \in U(\mathbb{Z}_{p^n})$$

where $b_K\colon \mathbb{Z}_{2^n}\times\mathbb{Z}_{2^n}\to\mathbb{Z}_{2^n}$ (respectively, $c_L\colon \mathbb{Z}_{p^n}\times\mathbb{Z}_{p^n}\to\mathbb{Z}_{p^n}$) is the $\mathbb{Z}_{2^n}$-bilinear ($\mathbb{Z}_{p^n}$-bilinear), non-degenerate, skew-symmetric map having coordinate matrix $\left[\begin{smallmatrix} 2^{n-1} & K \\ -K & 0 \end{smallmatrix}\right]\in M_2(\mathbb{Z}_{2^n})$ ($\left[\begin{smallmatrix} 0 & L \\ -L & 0 \end{smallmatrix}\right]\in M_2(\mathbb{Z}_{p^n})$).
Up to bicharacter equivalence, the definitions of u_{2^n}, w_{p^n} do not depend on the particular choice of primitive roots $\xi,\zeta\in\mathbb{T}$, respectively (or equivalently, on the choice of $K\in U(\mathbb{Z}_{2^n})$ and $L\in U(\mathbb{Z}_{p^n})$). Indeed, once fixed $K\in U(\mathbb{Z}_{2^n})$, $u\sim u_{2^n}$ if and only if

$$u(\mathbf{x},\mathbf{y}) = e^{i\frac{2\pi}{2^n}\mathbf{x}^t\left(T^t\left[\begin{smallmatrix} 2^{n-1} & K \\ -K & 0 \end{smallmatrix}\right]T\right)\mathbf{y}}$$

for some matrix $T\in GL_2(\mathbb{Z}_{2^n}) = \{M\in M_2(\mathbb{Z}_{2^n})\colon \det(M)\in U(\mathbb{Z}_{2^n})\}$, so that bicharacter equivalence translates into *congruence* (change of basis) of coordinate matrices over $\mathbb{Z}_{2^n}$. Then, for each $K'\in U(\mathbb{Z}_{2^n})$,

- if $T := \left[\begin{smallmatrix} K' & 0 \\ 0 & K^{-1} \end{smallmatrix}\right]\in GL_2(\mathbb{Z}_{2^n})$, then $T^t\left[\begin{smallmatrix} 2^{n-1} & K \\ -K & 0 \end{smallmatrix}\right]T = \left[\begin{smallmatrix} 2^{n-1} & K' \\ -K' & 0 \end{smallmatrix}\right]$
- if $T := \left[\begin{smallmatrix} 0 & K' \\ -K^{-1} & 0 \end{smallmatrix}\right]\in GL_2(\mathbb{Z}_{2^n})$, then $T^t\left[\begin{smallmatrix} 2^{n-1} & K \\ -K & 0 \end{smallmatrix}\right]T = \left[\begin{smallmatrix} 0 & K' \\ -K' & 2^{n-1} \end{smallmatrix}\right]$
- if $T := \left[\begin{smallmatrix} K & K' \\ -K^{-1} & 0 \end{smallmatrix}\right]\in GL_2(\mathbb{Z}_{2^n})$, then $T^t\left[\begin{smallmatrix} 2^{n-1} & K \\ -K & 0 \end{smallmatrix}\right]T = \left[\begin{smallmatrix} 2^{n-1} & 2^{n-1}+K' \\ -(2^{n-1}+K') & 2^{n-1} \end{smallmatrix}\right]$

In other words, $\left[\begin{smallmatrix} 2^{n-1} & K \\ -K & 0 \end{smallmatrix}\right]\sim\left[\begin{smallmatrix} 2^{n-1} & K' \\ -K' & 0 \end{smallmatrix}\right]\sim\left[\begin{smallmatrix} 0 & K' \\ -K' & 2^{n-1} \end{smallmatrix}\right]\sim\left[\begin{smallmatrix} 2^{n-1} & 2^{n-1}+K' \\ -(2^{n-1}+K') & 2^{n-1} \end{smallmatrix}\right]$ for any $K,K'\in U(\mathbb{Z}_{2^n})$. Analogously, once fixed $L\in U(\mathbb{Z}_{p^n})$, $u\sim w_{p^n}$ if and only if

$$u(\mathbf{x},\mathbf{y}) = e^{i\frac{2\pi}{p^n}\mathbf{x}^t\left(T^t\left[\begin{smallmatrix} 0 & L \\ -L & 0 \end{smallmatrix}\right]T\right)\mathbf{y}}$$

for some $T\in GL_2(\mathbb{Z}_{p^n})$. In particular, for every $L'\in U(\mathbb{Z}_{p^n})$, if $T := \left[\begin{smallmatrix} L' & 0 \\ 0 & L^{-1} \end{smallmatrix}\right]\in GL_2(\mathbb{Z}_{p^n})$, then $T^t\left[\begin{smallmatrix} 0 & L \\ -L & 0 \end{smallmatrix}\right]T = \left[\begin{smallmatrix} 0 & L' \\ -L' & 0 \end{smallmatrix}\right]$, so that $\left[\begin{smallmatrix} 0 & L \\ -L & 0 \end{smallmatrix}\right]\sim\left[\begin{smallmatrix} 0 & L' \\ -L' & 0 \end{smallmatrix}\right]$ for any $L,L'\in U(\mathbb{Z}_{p^n})$.
We also define the non-degenerate skew-symmetric bicharacter on the *Klein 4-group* $K_4 := \mathbb{Z}_2\times\mathbb{Z}_2$ as

$$u_{\mathtt{K}}(\mathbf{x},\mathbf{y}) := (-1)^{\mathbf{x}^t\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]\mathbf{y}} = (-1)^{\mathbf{x}^t\cdot\mathbf{y}},\quad \mathbf{x},\mathbf{y}\in K_4\,.$$

By the above discussion, $u_{\mathtt{K}}\sim u_2$ via $T = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. Moreover, $u_{\mathtt{K}}$ is evidently *reducible*, since $u_{\mathtt{K}}\sim u_{\mathtt{F}}\otimes u_{\mathtt{F}}$. We will refer to $u_{\mathtt{K}}$ as the *Klein* bicharacter of K_4. The results below are due to Zolotykh (Lemma 6 at p. 459, Lemma 7 at p. 460 and Theorem 1 at p. 461 in [82]).

Theorem III.5.1 (Zolotykh, [82])
The following facts hold true:

(i) The elementary bicharacters are pairwise non-equivalent: for every $m,n\in\mathbb{N}$ and prime $p\geq 2$, $u_{\mathtt{F}}\not\sim u_{2^m}\not\sim w_{p^n}$

(ii) $u_2\sim u_{\mathtt{K}}$ is the only reducible bicharacter among the elementary ones

(iii) conversely, if the pair $(\mathcal{G},u)$ is s.t. $\mathcal{G}\neq(0)$ and $u\in\mathsf{A}(\mathcal{G})$ is non-degenerate and irreducible, then $(\mathcal{G},u)$ is equivalent to $(\mathbb{Z}_2,u_{\mathtt{F}})$, $(\mathbb{Z}_{2^m}\times\mathbb{Z}_{2^m},u_{2^m})$ for unique $m\geq 2$, or $(\mathbb{Z}_{p^n},w_{p^n})$ for unique $n\geq 1$

(iv) $u_{2^n}\otimes u_{2^m}\sim w_{2^n}\otimes u_{2^m}$ for every $n\geq m\geq 1$

(v) $u_{\mathtt{F}}\otimes u_{2^m}\sim u_{\mathtt{F}}\otimes w_{2^m}$ for every $m\geq 1$

(vi) if $\mathcal{G}$ is a finite, abelian p-group ($|\mathcal{G}| = p^d$ for some prime $p\geq 2, d\in\mathbb{N}$) and $u\in\mathsf{A}(\mathcal{G})$ is non-degenerate, then there exists a unique (up to rearrangements) group presentation for which

$$(\mathcal{G},u)\sim\left(\bigtimes_{i=1}^{N}\mathcal{G}_i,\bigotimes_{i=1}^{N}u|_{\mathcal{G}_i}\right)$$

where $(\mathcal{G}_i, u|_{\mathcal{G}_i}) \sim (\mathbb{Z}_{p^{n_i}} \times \mathbb{Z}_{p^{n_i}}, w_{p^{n_i}})$ $(n_i \in \mathbb{N})$ for every $i = 1, \dots, N$, except at most one j, for which either $(\mathcal{G}_j, u|_{\mathcal{G}_j}) \sim (\mathbb{Z}_2, u_{\mathtt{F}})$ or $(\mathcal{G}_j, u|_{\mathcal{G}_j}) \sim (\mathbb{Z}_{2^m} \times \mathbb{Z}_{2^m}, u_{2^m})$ $(m \in \mathbb{N})$

(vii) if $\mathcal{G}$ is a finite, abelian group and $u \in \mathsf{A}(\mathcal{G})$ is non-degenerate, then there exists a unique (up to rearrangements) group presentation for which

$$(\mathcal{G}, u) \sim \left(\bigtimes_{i=1}^{N} \mathcal{G}_i, \bigotimes_{i=1}^{N} u|_{\mathcal{G}_i} \right)$$

where each $\mathcal{G}_i$ is a p_i-group, with the p_i's pairwise coprime, and $u|_{\mathcal{G}_i} \in \mathsf{A}(\mathcal{G}_i)$ is non-degenerate $(i = 1, \dots, n)$. If there exists (unique) $j = 1, \dots, N$ s.t. $\mathcal{G}_j$ is a 2-group and its decomposition in point (vi) contains a (unique) Fermi factor $(\mathbb{Z}_2, u_{\mathtt{F}})$, then $(\mathcal{G}, u) \sim (\mathbb{Z}_2 \times \Delta_+, u_{\mathtt{F}} \otimes u|_{\Delta_+})$; otherwise $\mathcal{G}$ is *of central type*, i.e. it admits a non-degenerate (normalized) 2-cocycle/multiplier $\omega \in Z^2(\mathcal{G}, \mathbb{T})$.

Remark III.5.2
Following a well-known Scheunert's construction (see [72]), every $u \in \mathsf{A}(\mathcal{G})$ gives rise to some $\widetilde{u} \in \Lambda(\mathcal{G})$. Indeed, let $u_0 \in \mathsf{A}(\mathcal{G})$ be the bicharacter defined by $u_0(g, g') := \begin{cases} -1 \text{ if } (g, g') \in \Delta_- \times \Delta_- \\ +1 \text{ otherwise} \end{cases}$.
Then, it is easy to verify that $\widetilde{u} := u_0 u \in \Lambda(\mathcal{G}) \subseteq \mathsf{A}(\mathcal{G})$. In particular, if $\mathcal{G}$ is a finitely generated abelian group, then (by Lemma 2 in Section 5 and Theorem 2 in Section 6 of [72]) there exists a well-defined epimorphism between the 2-cohomology group $H^2(\mathcal{G}, \mathbb{T})$ of $\mathcal{G}$ and the group of alternating bicharacters $\Lambda(\mathcal{G})$ on $\mathcal{G}$

$$\begin{aligned} \mathsf{Alt}\colon H^2(\mathcal{G}, \mathbb{T}) &\twoheadrightarrow \Lambda(\mathcal{G}) \\ [\omega] &\mapsto \big[(g, g') \mapsto \omega(g, g') \overline{\omega(g', g)} \big] \end{aligned}$$

in which case $\widetilde{u} = \mathsf{Alt}([\omega])$ for some representative $\omega \in Z^2(\mathcal{G}, \mathbb{T})$ (in particular, ω can be chosen in $\mathsf{B}(\mathcal{G})$). For instance,

- $u_0 u_{\mathtt{F}} = \mathbb{1}_{\mathbb{Z}_2 \times \mathbb{Z}_2} = \mathsf{Alt}([\mathbb{1}_{\mathbb{Z}_2 \times \mathbb{Z}_2}])$
- $u_0 u_{2^n} \sim w_{2^n}$, for every $n \in \mathbb{N}$, and $w_{2^n} = \mathsf{Alt}([\omega])$, where $\omega(\mathbf{x}, \mathbf{y}) := \zeta^{\mathbf{x}^t \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] \mathbf{y}}$, $\mathbf{x}, \mathbf{y} \in \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n}$, $\zeta \in \mathbb{T}$ primitive 2^n-th root of unity. In particular, for $n = 1$, $u_0 u_{\mathtt{K}} \sim w_2$.

III.6. The action of $\mathfrak{S}$ on a twisted chain

From now on, we will deal with non-degenerate $u \in \mathsf{A}(\widehat{G})$ only. In a nutshell, as exposed in the previous section, the reason of this assumption is that in such a case the flip map Φ_u realizes an involutive *-automorphism of $\mathfrak{B} \textcircled{u} \mathfrak{B}$ which intertwines the product action of $G \times G$, and *a fortiori* the diagonal action of G.
For each $n \in \mathbb{N}$, define the discrete segment $\mathbf{n} := [1, n] = \{1, \dots, n\}$, the symmetric group $\mathfrak{S}_{\mathbf{n}}$ consisting of the $n!$ permutations of $\mathbf{n}$, and the $n-1$ adjacent transpositions $\pi_i := (i\ i{+}1) \in \mathfrak{S}_{\mathbf{n}}$ $(i = 1, \dots, n-1)$. Then, the $\mathbb{C}$-linear extension of the map

$$\pi_i(b_1 \textcircled{u} \cdots \textcircled{u} b_i \textcircled{u} b_{i+1} \textcircled{u} \cdots \textcircled{u} b_n) := u(b_i, b_{i+1})\, b_1 \textcircled{u} \cdots \textcircled{u} b_{i+1} \textcircled{u} b_i \textcircled{u} \cdots \textcircled{u} b_n\,,$$

defined on the elementary tensor products of homogeneous elements, isometrically extends to an element of $\mathsf{Aut}(\mathfrak{A}_n)$. This is just a particular case of a more general fact, as the following proposition shows. As a premise, fix $\rho \in \mathfrak{S}_{\mathbf{n}}$ and let $\mathcal{I}_\rho := \{(l, k) \in \mathbf{n} \times \mathbf{n} \colon l < k, \rho(l) > \rho(k)\}$

be the set of *inversions of* ρ. Its cardinality is $\mathsf{inv}(\rho) := |\mathcal{I}_\rho| \in \left\{0, \dots, \binom{n}{2}\right\}$, which coincides with the Kendall τ-distance $K_d(\rho, \mathrm{id}_{\mathbf{n}})$ of ρ from the trivial permutation $\mathrm{id}_{\mathbf{n}}$. Also notice that if $\Sigma\colon \mathbf{n}\times\mathbf{n} \to \mathbf{n}\times\mathbf{n}$ is the switch bijection, $\mathcal{I}_{\rho^{-1}} = (\rho\times\rho)\circ\Sigma(\mathcal{I}_\rho)$. We then have the following

Proposition III.6.1
Each $\rho \in \mathfrak{S}_{\mathbf{n}}$ induces an element of $\mathsf{Aut}(\mathfrak{A}_n)$. Moreover, for every homogeneous $b_l \in \mathfrak{B}_o$,

$$\rho(b_1 \textcircled{u} \cdots \textcircled{u}\, b_n) = \prod_{(l,k)\in\mathcal{I}_{\rho^{-1}}} u(b_l, b_k)\, b_{\rho(1)} \textcircled{u} \cdots \textcircled{u}\, b_{\rho(n)}\,. \tag{III.8}$$

Proof.
Since the $n-1$ adjacent transpositions of $\mathbf{n}$ form a generating set of the group $\mathfrak{S}_{\mathbf{n}} \sim S_n$, by the previous discussion, each $\rho \in \mathfrak{S}_{\mathbf{n}}$ induces an element of $\mathsf{Aut}(\mathfrak{A}_n)$. As concerns the formula, it suffices to prove it for any product ρ of N adjacent transpositions, with $N \geq 1$. We shall do it by induction on N. For $N = 1$, i.e. $\rho = \pi_i = (i\ i+1)$ for some $i = 1, \dots, n-1$, $\mathcal{I}_{\rho^{-1}} = \mathcal{I}_\rho = \{(i, i+1)\}$ and the formula reduces to the one exposed above. Let us suppose that the result holds for some $N \geq 1$ and prove it for $N+1$. If ρ is a product of N adjacent transpositions, then for every $i = 1, \dots, n-1$, by inductive hypothesis we have

$$\begin{aligned}(\pi_i\circ\rho)(b_1 \textcircled{u} \cdots \textcircled{u}\, b_n) &= \\ &= \prod_{(l,k)\in\mathcal{I}_{\rho^{-1}}} u(b_l, b_k)\, u(b_{\rho(i)}, b_{\rho(i+1)})(b_{\rho(1)} \textcircled{u} \cdots \textcircled{u}\, b_{\rho(i+1)} \textcircled{u}\, b_{\rho(i)} \textcircled{u} \cdots \textcircled{u}\, b_{\rho(n)})\,.\end{aligned}$$

Here, we are using the standard convention of reading the composition of cycles *from right to left*, as for general functions. We are done if we prove the following equality:

$$u(b_{\rho(i)}, b_{\rho(i+1)}) \prod_{(l,k)\in\mathcal{I}_{\rho^{-1}}} u(b_l, b_k) = \prod_{(x,y)\in\mathcal{I}_{(\rho\circ\pi_i)^{-1}}} u(b_x, b_y)$$

or, equivalently,

$$u(b_{\rho(i)}, b_{\rho(i+1)}) \prod_{(l,k)\in\mathcal{I}_{\rho^{-1}}} u(b_l, b_k) \prod_{(x,y)\in\mathcal{I}_{\pi_i\circ\rho^{-1}}} \overline{u(b_x, b_y)} = 1\,. \tag{III.9}$$

(Recall that a composition of two permutations on an elementary tensor product acts on the indices by reversing the composition).
Firstly, suppose $\rho(i) < \rho(i+1)$. Then, $(x, y) \in \mathcal{I}_{\pi_i\circ\rho^{-1}}$ if and only if either $(x, y) = (\rho(i), \rho(i+1))$ or $(x, y) \in \mathcal{I}_{\rho^{-1}}$ and satisfies one of the following seven cases:

(1) $\rho^{-1}(y) < \rho^{-1}(x) < i$

(2) $\rho^{-1}(y) < i < i+1 = \rho^{-1}(x)$

(3) $\rho^{-1}(y) < i < i+1 < \rho^{-1}(x)$

(4) $\rho^{-1}(y) = i < i+1 < \rho^{-1}(x)$

(5) $\rho^{-1}(y) = i+1 < \rho^{-1}(x)$

(6) $i+1 < \rho^{-1}(y) < \rho^{-1}(x)$

(7) $\rho^{-1}(y) < i = \rho^{-1}(x)$

In other words, $\mathcal{I}_{\pi_i\circ\rho^{-1}} = \mathcal{I}_{\rho^{-1}} \dot\cup \{(\rho(i), \rho(i+1))\}$ and Equation III.9 follows. Instead, if $\rho(i+1) < \rho(i)$, then $\mathcal{I}_{\rho^{-1}} = \mathcal{I}_{\pi_i\circ\rho^{-1}} \dot\cup \{(\rho(i+1), \rho(i))\}$ and again Equation III.9 is satisfied. □

For each $n \in \mathbb{N}$, if $\jmath_n\colon \mathfrak{A}_n \hookrightarrow \mathfrak{A}$ is the canonical embedding of $\mathfrak{A}_n$ into $\mathfrak{A}$ and $\rho \in \mathfrak{S}_{\mathbf{n}}$, $\jmath_n\circ\rho\colon \mathfrak{A}_n \to \mathfrak{A}$ extends to a well-defined *-automorphism of $\mathfrak{A}$ by universal property of the

C^*-inductive limit. We can say more. Let $\mathfrak{S}$ be the *finitary* symmetric group on the set $\mathbb{N}$, i.e. the group of permutations of $\mathbb{N}$ leaving fixed all but a *finite* amount of elements (it is a normal, conjugacy-closed subgroup of the symmetric group $S_{\mathbb{N}}$ on $\mathbb{N}$). It can be built from scratch as the direct limit of the system $(\mathfrak{S}_{\mathbf{n}}, \phi_{nm})_{n\leq m}$, where the connecting map $\phi_{nm}\colon \mathfrak{S}_{\mathbf{n}} \hookrightarrow \mathfrak{S}_{\mathbf{m}}$ embeds each $\rho \in \mathfrak{S}_{\mathbf{n}}$ into $\mathfrak{S}_{\mathbf{m}}$ as the unique permutation of $\mathbf{m}$ which act as ρ on the first n elements and leaves the elements in $\mathbf{m}\setminus\mathbf{n} = \{n+1,\dots,m\}$ fixed. The groups $\mathfrak{S}_{\mathbf{n}}$ are then canonically embedded in $\mathfrak{S}$ and form an ascending chain of subgroups of $\mathfrak{S}$. Explicitly,

$$\mathfrak{S} = \varinjlim_n \mathfrak{S}_{\mathbf{n}} = \bigcup_{n\in\mathbb{N}} \mathfrak{S}_{\mathbf{n}}\,.$$

Now, there exists a unique group representation of $\mathfrak{S}$ on $\mathfrak{A}$. Indeed, for each $i \in \mathbb{N}$, let $\Phi_i \in \mathsf{Aut}(\mathfrak{A})$ be the isometric extension to $\mathfrak{A}$ of the adjacent transposition $\pi_i \in \mathsf{Aut}(\mathfrak{A}_{i+1})$. In particular, Φ_i is involutive. Firstly, there exists a unique representation of the free (non-abelian) group $\mathbb{F}$ over $\mathbb{N}$

$$\begin{aligned}\Pi\colon \mathbb{F} &\to \mathsf{Aut}(\mathfrak{A})\\ w &\mapsto \Phi_{i_1}\circ\cdots\circ\Phi_{i_n}\end{aligned}$$

where $i_1\cdots i_n$ is the (unique) reduced form of the word $w\in\mathbb{F}$. (Since the Φ_i's are involutive, this group representation is not faithful.) Secondly, we have the following

Theorem III.6.2
Let $\rho \in \mathfrak{S}$ expressed (not uniquely) as a finite product of adjacent transpositions $\rho = \pi_{i_1}\pi_{i_2}\cdots\pi_{i_n}$, $(i_k \in \mathbb{N},\ k = 1,\dots,n)$. Then, the assignment $\rho \mapsto \alpha_\rho := \Phi_{i_i}\circ\cdots\circ\Phi_{i_n}$ realizes a well-defined, pointwise norm-continuous action $\mathfrak{S} \overset{\alpha}{\curvearrowright} \mathfrak{A}$. Moreover, α commutes with the diagonal action $\delta^{(\beta)}$ of G on $\mathfrak{A}$: $\delta^{(\beta)}\circ\alpha = \alpha\circ\delta^{(\beta)}$. The action α is faithful provided that $\mathfrak{B}\neq\mathbb{C}$.

Proof.
It is well-known that $\mathfrak{S} \cong (\mathbb{F}\,|\,R)$ where R is the set of relations in $\mathbb{F}$

$$\begin{cases} i_n^2 = 1\\ i_n i_{n+k} = i_{n+k} i_n \ (k\geq 2)\\ i_n i_{n+1} i_n = i_{n+1} i_n i_{n+1}\end{cases}$$

for every $n\in\mathbb{N}$. (This is the so-called Coxeter presentation of the finitary symmetric group $\mathfrak{S}$). Since the normal subgroup $N\subset\mathbb{F}$ generated by the relations above lies in $\ker(\Pi)$, then Π passes to the quotient modulo N, yielding a well-defined representation $\widetilde{\Pi}$ of $\mathfrak{S}$ on $\mathfrak{A}$. It is easy to see that $\widetilde{\Pi} = \alpha$. Indeed, we already observed that the Φ_i are involutive and, easily, $\Phi_i\circ\Phi_{i+k} = \Phi_{i+k}\circ\Phi_i$ for every $k\geq 2$. Lastly, the third relation is guaranteed by the Yang-Baxter equality, satisfied by Φ_u on $\mathfrak{B}_o \mathbin{\textcircled{u}} \mathfrak{B}_o \mathbin{\textcircled{u}} \mathfrak{B}_o$

$$(\Phi_u \mathbin{\textcircled{u}} I_{\mathfrak{B}_o})\circ(I_{\mathfrak{B}_o}\mathbin{\textcircled{u}}\Phi_u)\circ(\Phi_u\mathbin{\textcircled{u}} I_{\mathfrak{B}_o}) = (I_{\mathfrak{B}_o}\mathbin{\textcircled{u}}\Phi_u)\circ(\Phi_u\mathbin{\textcircled{u}} I_{\mathfrak{B}_o})\circ(I_{\mathfrak{B}_o}\mathbin{\textcircled{u}}\Phi_u)\,,$$

then extended to $\mathfrak{B}\mathbin{\textcircled{u}}\mathfrak{B}\mathbin{\textcircled{u}}\mathfrak{B}$. Since $\mathfrak{S}$ is discrete, α is clearly pointwise norm-continuous. That α commutes with $\delta^{(\beta)}$ is easily verifiable on the total set of localized homogeneous elements of $\mathfrak{A}$. Lastly, if $\mathfrak{B}\neq\mathbb{C}$, α is also faithful. Indeed, if $\rho\in\mathfrak{S}\setminus\{\mathrm{id}_{\mathbb{N}}\}$, there exist $n,k\in\mathbb{N}$ s.t. $k\leq n$, $\rho\in\mathfrak{S}_{\mathbf{n}}$ and $\rho(k)\neq k$. Let $b_k\in\mathfrak{B}\setminus\mathbb{C}\mathbb{1}_{\mathfrak{B}}$ be homogeneous and $b_j = \mathbb{1}_{\mathfrak{B}}$ for every $j\neq k$. Then, by (III.8),

$$\begin{aligned}\alpha_\rho(b_0\mathbin{\textcircled{u}}\cdots\mathbin{\textcircled{u}} b_k\mathbin{\textcircled{u}}\cdots\mathbin{\textcircled{u}} b_n\mathbin{\textcircled{u}}\cdots) &= \alpha_\rho(\mathbb{1}_{\mathfrak{B}}\mathbin{\textcircled{u}}\cdots\mathbin{\textcircled{u}} b_k\mathbin{\textcircled{u}}\cdots\mathbin{\textcircled{u}}\mathbb{1}_{\mathfrak{B}}\mathbin{\textcircled{u}}\cdots) =\\ &= \rho(\mathbb{1}_{\mathfrak{B}}\mathbin{\textcircled{u}}\cdots\mathbin{\textcircled{u}} b_k\mathbin{\textcircled{u}}\cdots\mathbin{\textcircled{u}}\mathbb{1}_{\mathfrak{B}})\mathbin{\textcircled{u}}\mathbb{1}_{\mathfrak{B}}\mathbin{\textcircled{u}}\cdots =\\ &= (\mathbb{1}_{\mathfrak{B}}\mathbin{\textcircled{u}}\cdots\mathbin{\textcircled{u}}\underbrace{\mathbb{1}_{\mathfrak{B}}}_{k^{\text{th}}-\text{place}}\mathbin{\textcircled{u}}\cdots\mathbin{\textcircled{u}}\underbrace{b_k}_{\rho^{-1}(k)^{\text{th}}-\text{place}}\mathbin{\textcircled{u}}\cdots\mathbin{\textcircled{u}}\mathbb{1}_{\mathfrak{B}})\mathbin{\textcircled{u}}\mathbb{1}_{\mathfrak{B}}\mathbin{\textcircled{u}}\cdots\\ &\neq b_0\mathbin{\textcircled{u}}\cdots\mathbin{\textcircled{u}} b_k\mathbin{\textcircled{u}}\cdots\mathbin{\textcircled{u}} b_n\mathbin{\textcircled{u}}\cdots\end{aligned}$$

i.e. $\alpha_\rho \neq \mathrm{id}_{\mathfrak{A}}$. $\square$

Theorem III.6.2 yields a new C^*-system $(\mathfrak{A}, \mathfrak{S}, \gamma)$. We conclude this section by reporting and demonstrating a combinatorial estimate which will be of primary importance in the investigation of the ergodic theory of the C^*-system $(\mathfrak{A}, \mathfrak{S}, \alpha)$. In words, it essentially states that the non-disjoining permutations in $\mathfrak{S}$ are quite "rare". For completeness, we give this "rareness" result also for the family $\mathcal{A}$ of finite *even* permutations of $\mathbb{N}$, the finitary alternating group. It is a simple, normal, index 2 subgroup of $\mathfrak{S}$.

Lemma III.6.3 (Combinatorial estimate for non-disjoining permutations)
Let $m, n \geq 1$. For sufficiently large N, there exists some positive constant $C_{m,n} > 0$ (depending on m, n only) such that

$$\frac{|\{\rho \in \mathfrak{S}_{\mathbf{N}} \colon \mathbf{m} \cap \rho(\mathbf{n}) \neq \varnothing\}|}{(N-1)!} = 2\,\frac{|\{\rho \in \mathcal{A}_{\mathbf{N}} \colon \mathbf{m} \cap \rho(\mathbf{n}) \neq \varnothing\}|}{(N-1)!} \leq C_{m,n}\,.$$

Proof.
For each $N \geq m+n$, set $\mathcal{D}_{N,m,n} := |\{\rho \in \mathfrak{S}_{\mathbf{N}} \colon \mathbf{m} \cap \rho(\mathbf{n}) = \varnothing\}| = \dfrac{(N-m)!(N-n)!}{(N-(m+n))!}$. By Stirling's approximation formula, when N is sufficiently large,

$$\frac{\mathcal{D}_{N,m,n}}{N!} \sim \sqrt{\frac{(N-m)(N-n)}{N(N-m-n)}}e^{\lambda} = \sqrt{\frac{\left(1-\frac{m}{N}\right)\left(1-\frac{n}{N}\right)}{1-\frac{m+n}{N}}}e^{\lambda_{N,m,n}}$$

where

$$\begin{aligned}
\lambda_{N,m,n} &:= (N-m)\log(N-m)+(N-n)\log(N-n)-N\log(N)-(N-m-n)\log(N-m-n) = \\
&= N\left[\left(1-\frac{m}{N}\right)\log\left(1-\frac{m}{N}\right)+\left(1-\frac{n}{N}\right)\log\left(1-\frac{n}{N}\right)-\left(1-\frac{m+n}{N}\right)\log\left(1-\frac{m+n}{N}\right)\right] = \\
&= -m\left(1-\frac{m}{N}\right)-n\left(1-\frac{n}{N}\right)+(m+n)\left(1-\frac{m+n}{N}\right)+o(1) = -\frac{2mn}{N}+o(1)
\end{aligned}$$

after performing a 1^{st}-order MacLaurin expansion around N^{-1}. It follows that

$$\frac{\mathcal{D}_{N,m,n}}{N!} = \left(1-\frac{m}{2N}\right)\left(1-\frac{n}{2N}\right)\left(1+\frac{m+n}{2N}\right)\left(1-\frac{2mn}{N}\right)+o\left(N^{-4}\right) = 1-\frac{2mn}{N}+o(N^{-2})$$

whence

$$\frac{|\{\rho \in \mathfrak{S}_{\mathbf{N}} \colon \mathbf{m} \cap \rho(\mathbf{n}) \neq \varnothing\}|}{N!} = 1-\frac{\mathcal{D}_{N,m,n}}{N!} = \frac{2mn}{N}+o(N^{-2})\,.$$

In particular, there exists $C_{m,n} > 0$ such that

$$\frac{|\{\rho \in \mathfrak{S}_{\mathbf{N}} \colon \mathbf{m} \cap \rho(\mathbf{n}) \neq \varnothing\}|}{(N-1)!} \leq C_{m,n}\,.$$

Lastly, we see at once that $|\{\rho \in \mathcal{A}_{\mathbf{N}} \colon \mathbf{m} \cap \rho(\mathbf{n}) = \varnothing\}| = \dfrac{\frac{(N-m)!}{2}\,\frac{(N-n)!}{2}}{\frac{(N-(m+n))!}{2}} = \dfrac{\mathcal{D}_{N,m,n}}{2}$, so that

$$|\{\rho \in \mathcal{A}_{\mathbf{N}} \colon \mathbf{m} \cap \rho(\mathbf{n}) \neq \varnothing\}| = \frac{N! - \mathcal{D}_{N,m,n}}{2}\,,$$

and the proof is accomplished. $\square$

III.7. Ergodic theory of $(\mathfrak{A}, \mathfrak{S}, \alpha)$

A state $\omega \in \mathcal{S}(\mathfrak{A})$ is called *symmetric* if it is $\mathfrak{S}$-invariant, that is $\omega \circ \alpha_\rho = \omega$ for every $\rho \in \mathfrak{S}$. As seen in Section III.2, the family $\mathcal{S}_{\mathfrak{S}}(\mathfrak{A})$ of the symmetric states is a convex and weakly-* compact subset of $\mathcal{S}(\mathfrak{A})$, with extremal points forming $\mathcal{E}_{\mathfrak{S}}(\mathfrak{A}) \neq \varnothing$, the family of the ($\mathfrak{S}$-)*ergodic* states. We also remind that, for each $\omega \in \mathcal{S}_{\mathfrak{S}}(\mathfrak{A})$, the *compression* (or *corner*) map

$$\begin{aligned} E_\omega \mathcal{B}(\mathcal{H}_\omega) E_\omega &\to \mathcal{B}(\mathcal{H}_\omega^{\mathfrak{S}}) \\ E_\omega X E_\omega &\mapsto X^{\mathfrak{S}} := E_\omega X|_{\mathcal{H}_\omega^{\mathfrak{S}}} \end{aligned}$$

is a *-isomorphism of C^*-algebras, identifying $E_\omega \pi_\omega(\mathfrak{A}) E_\omega$ with an operator system $\pi_\omega(\mathfrak{A})^{\mathfrak{S}} \subseteq \mathcal{B}(\mathcal{H}_\omega^{\mathfrak{S}})$ acting upon $\mathcal{H}_\omega^{\mathfrak{S}}$. Observe that, for $x \in \mathfrak{A}$, $\pi_\omega(x)^{\mathfrak{S}} = 0$ if and only if $\pi_\omega(x)\mathcal{H}_\omega^{\mathfrak{S}} \subseteq (\mathcal{H}_\omega^{\mathfrak{S}})^\perp$.

Theorem III.7.1 (Commutation and Anticommutation Relations in $\pi_\omega(\mathfrak{A})^{\mathfrak{S}}$)
Let $\omega \in \mathcal{S}_{\mathfrak{S}}(\mathfrak{A})$ and $x, y, a, b \in \mathfrak{A}$ homogeneous. Then,

(i) $\left\{\pi_\omega(x)^{\mathfrak{S}}, \pi_\omega(y)^{\mathfrak{S}}\right\} = \big(1 + \overline{u(x,y)}\big)\pi_\omega(x)^{\mathfrak{S}}\pi_\omega(y)^{\mathfrak{S}}$

(ii) $\left[\pi_\omega(x)^{\mathfrak{S}}, \pi_\omega(y)^{\mathfrak{S}}\right] = \big(1 - \overline{u(x,y)}\big)\pi_\omega(x)^{\mathfrak{S}}\pi_\omega(y)^{\mathfrak{S}}$

(iii) $\mathcal{M}\{\omega(a[x, \rho(y)]b)\} = (1 - \overline{u(x,y)})\, \overline{u(b,y)} \left\langle \pi_\omega(axb)^{\mathfrak{S}}\pi_\omega(y)^{\mathfrak{S}}\xi_\omega, \xi_\omega \right\rangle_{\mathcal{H}_\omega^{\mathfrak{S}}}$

In particular, if $\pi_\omega(x)^{\mathfrak{S}} \in \mathcal{B}(\mathcal{H}_\omega^{\mathfrak{S}})$ is non-zero, then $u(x,x) = 1$ and $\pi_\omega(x)^{\mathfrak{S}}$ is normal.

Proof.
We start from homogeneous $x, y \in \mathfrak{A}_\infty$, respectively localized in the discrete segments $\mathbf{t} := [1, t]$ and $\mathbf{u} := [1, u]$ $(t, u \in \mathbb{N})$. By the celebrated von Neumann ergodic theorem,

$$\{E_\omega \pi_\omega(x) E_\omega, E_\omega \pi_\omega(y) E_\omega\} = \mathcal{M}\{E_\omega \pi_\omega(\{x, \rho(y)\}) E_\omega\} = \underset{N \to +\infty}{s\text{-lim}} \frac{1}{N!} \sum_{\rho \in \mathfrak{S}_{\mathbf{N}}} E_\omega \pi_\omega(\{x, \rho(y)\}) E_\omega\,.$$

Let $v := \max\{t, u\}$. For each $N \geq 2v$, consider the family $\Gamma_{N,v} := \{g \in \mathfrak{S}_{\mathbf{N}} \mid \mathbf{v} \cap \rho(\mathbf{v}) = \varnothing\}$ of permutations of $\mathbf{N} := [1, N]$ which fully displace $\mathbf{v}$. Then,

$$\begin{aligned} \sum_{\rho \in \mathfrak{S}_{\mathbf{N}}} E_\omega \pi_\omega\big(\{x, \rho(y)\}\big) E_\omega &= \sum_{\rho \in \Gamma_{N,v}} E_\omega \pi_\omega\big(\{x, \rho(y)\}\big) E_\omega + \sum_{\rho \notin \Gamma_{N,v}} E_\omega \pi_\omega\big(\{x, \rho(y)\}\big) E_\omega = \\ &= \big(1 + \overline{u(x,y)}\big) \sum_{\rho \in \Gamma_{N,v}} E_\omega \pi_\omega\big(x\rho(y)\big) E_\omega + \sum_{\rho \notin \Gamma_{N,v}} E_\omega \pi_\omega\big(\{x, \rho(y)\}\big) E_\omega = \\ = \big(1+\overline{u(x,y)}\big) \sum_{\rho \in \mathfrak{S}_{\mathbf{N}}} E_\omega \pi_\omega\big(x\rho(y)\big) E_\omega &+ \sum_{\rho \notin \Gamma_{N,v}} \Big(E_\omega \pi_\omega\big(\{x, \rho(y)\}\big) E_\omega - \big(1+\overline{u(x,y)}\big) E_\omega \pi_\omega\big(x\rho(y)\big) E_\omega\Big) = \\ &= \big(1 + \overline{u(x,y)}\big) \sum_{\rho \in \mathfrak{S}_{\mathbf{N}}} E_\omega \pi_\omega\big(x\rho(y)\big) E_\omega + \sum_{\rho \notin \Gamma_{N,v}} E_\omega \pi_\omega\big(\rho(y)x - \overline{u(x,y)}x\rho(y)\big) E_\omega \end{aligned}$$

By Lemma III.6.3, the norm of the second addendum above is $o(N!)$ as $N \geq 2v$ tends to $+\infty$:

$$\begin{aligned} \frac{1}{N!} \left\| \sum_{\rho \notin \Gamma_{N,v}} E_\omega \pi_\omega\big(\rho(y)x - \overline{u(x,y)}x\rho(y)\big) E_\omega \right\| &\leq \frac{1}{N!} \sum_{\rho \notin \Gamma_{N,v}} \left\|\rho(y)x - \overline{u(x,y)}x\rho(y)\right\| \\ &\leq 2\|x\|\|y\| \frac{|\Gamma_{N,v}^c|}{N!} \leq 2\|x\|\|y\| \frac{C_v}{N} \xrightarrow{N \uparrow +\infty} 0\,. \end{aligned}$$

Therefore, by passing to the limit as N tends to $+\infty$:

$$\{E_\omega\pi_\omega(x)E_\omega, E_\omega\pi_\omega(y)E_\omega\} = \big(1+\overline{u(x,y)}\big) \operatorname*{s\text{-lim}}_{N\to+\infty} \frac{1}{N!}\sum_{\rho\in\mathfrak{S}_\mathbf{N}} E_\omega\pi_\omega(x\rho(y))E_\omega =$$
$$= \big(1+\overline{u(x,y)}\big) \operatorname*{s\text{-lim}}_{N\to+\infty} \frac{1}{N!}\sum_{\rho\in\mathfrak{S}_\mathbf{N}} E_\omega\pi_\omega(x)U_\omega(\rho)\pi_\omega(y)E_\omega = \big(1+\overline{u(x,y)}\big)E_\omega\pi_\omega(x)E_\omega\pi_\omega(y)E_\omega$$

and equality (i) is established for localized homogeneous elements $x,y\in\mathfrak{A}_\infty$. Now, since for every $\sigma\in\widehat{G}$, $\mathfrak{A}_\sigma=\overline{(\mathfrak{A}_\infty)_\sigma}^{\mathfrak{A}}$, if $x,y\in\mathfrak{A}$ are any pair of homogeneous elements and $\varepsilon\in(0,1)$, there exist $x_\varepsilon, y_\varepsilon\in\mathfrak{A}_\infty$ s.t. $\partial x_\varepsilon=\partial x$, $\partial y_\varepsilon=\partial y$ and $\|x-x_\varepsilon\|_\mathfrak{A}, \|y-y_\varepsilon\|_\mathfrak{A}<\varepsilon$. It follows that

$$\|E_\omega\pi_\omega(x)E_\omega\pi_\omega(y)E_\omega - E_\omega\pi_\omega(x_\varepsilon)E_\omega\pi_\omega(y_\varepsilon)E_\omega\| < \varepsilon(\|x\|+\|y\|+\varepsilon)$$
$$\|\{E_\omega\pi_\omega(x)E_\omega, E_\omega\pi_\omega(y)E_\omega\} - \{E_\omega\pi_\omega(x_\varepsilon)E_\omega, E_\omega\pi_\omega(y_\varepsilon)E_\omega\}\| < 2\varepsilon(\|x\|+\|y\|+\varepsilon)\,.$$

By arbitrariness of $\varepsilon\in(0,1)$, equality (i) is then established for any homogeneous elements $x,y\in\mathfrak{A}$. A similar proof holds for (ii).
As concerns (iii), firstly observe that for every $a,b,x,y\in\mathfrak{A}$, both $\mathcal{M}\{\omega(axb\rho(y))\}$ and $\mathcal{M}\{\omega(\rho(y)axb)\}$ are always perfectly meaningful, as

$$\mathcal{M}\{\omega(axb\rho(y))\} = \langle\pi_\omega(axb)E_\omega\pi_\omega(y)\xi_\omega,\xi_\omega\rangle\,,$$
$$\mathcal{M}\{\omega(\rho(y)axb)\} = \langle\pi_\omega(y)E_\omega\pi_\omega(axb)\xi_\omega,\xi_\omega\rangle\,.$$

Now, if a,b,x,y are homogeneous and belonging to $\mathfrak{A}_\infty$, by reasoning as above we get that $\mathcal{M}\{\omega(ax\rho(y)b)\}$ exists too, as

$$\mathcal{M}\{\omega(ax\rho(y)b)\} = \overline{u(b,y)}\mathcal{M}\{\omega(axb\rho(y))\}\,. \qquad \text{(III.10)}$$

To extend Equation III.10 to every homogeneous element in $\mathfrak{A}$, we observe that for any $\varepsilon\in(0,1)$, there exist $a_\varepsilon,x_\varepsilon,y_\varepsilon,b_\varepsilon\in\mathfrak{A}_\infty$ s.t. $\partial a_\varepsilon=\partial a, \partial x_\varepsilon=\partial x, \partial y_\varepsilon=\partial y, \partial b_\varepsilon=\partial b$ and $|\omega(ax\rho(y)b)-\omega(a_\varepsilon x_\varepsilon\rho(y_\varepsilon)b_\varepsilon)|<\varepsilon$. In particular, $\left|\frac{1}{N!}\sum_{\rho\in\mathfrak{S}_\mathbf{N}}\omega(ax\rho(y)b)-\frac{1}{N!}\sum_{\rho\in\mathfrak{S}_\mathbf{N}}\omega(a_\varepsilon x_\varepsilon\rho(y_\varepsilon)b_\varepsilon)\right|<\varepsilon$, $N\geq 1$.
Since $\left\{\frac{1}{N!}\sum_{\rho\in\mathfrak{S}_\mathbf{N}}\omega(a_\varepsilon x_\varepsilon\rho(y_\varepsilon)b_\varepsilon)\right\}_N$ converges, $\left\{\frac{1}{N!}\sum_{\rho\in\mathfrak{S}_\mathbf{N}}\omega(ax\rho(y)b)\right\}_N$ is a Cauchy sequence in $\mathbb{C}$, thus convergent as well.
It means that $\mathcal{M}\{\omega(ax\rho(y)b)\}$ exists and $\mathcal{M}\{\omega(ax\rho(y)b)\}=\lim_{\varepsilon\downarrow 0^+}\mathcal{M}\{\omega(a_\varepsilon x_\varepsilon\rho(y_\varepsilon)b_\varepsilon)$. By analogously approximating $\mathcal{M}\{\omega(axb\rho(y))\}$, we get Equation III.10 for any homogeneous elements $a,x,y,b\in\mathfrak{A}$. Similarly,

$$\mathcal{M}\{\omega(a\rho(y)xb)\} = u(a,y)\mathcal{M}\{\omega(\rho(y)axb)\} \qquad \text{(III.11)}$$

By Equation III.10 and Equation III.11,

$$\mathcal{M}\{\omega(ax\rho(y)b)\} = \overline{u(b,y)}\mathcal{M}\{\omega(axb\rho(y))\} = \overline{u(b,y)}\langle\pi_\omega(axb)E_\omega\pi_\omega(y)\xi_\omega,\xi_\omega\rangle$$
$$\mathcal{M}\{\omega(a\rho(y)xb)\} = u(a,y)\mathcal{M}\{\omega(\rho(y)axb)\} = \overline{u(x,y)}\,\overline{u(b,y)}\langle\pi_\omega(axb)E_\omega\pi_\omega(y)\xi_\omega,\xi_\omega\rangle$$

where we used point (i). Therefore, $\mathcal{M}\{\omega(a[x,\rho(y)]b)\} = (1-\overline{u(x,y)})\,\overline{u(b,y)}\langle\pi_\omega(axb)E_\omega\pi_\omega(y)\xi_\omega,\xi_\omega\rangle$, that is (iii).
Lastly, by exploiting (i) or (ii) with $y:=x^*$, we get

$$E_\omega\pi_\omega(x^*)E_\omega\pi_\omega(x)E_\omega = u(x,x)E_\omega\pi_\omega(x)E_\omega\pi_\omega(x^*)E_\omega\,.$$

If $E_\omega\pi_\omega(x)E_\omega$ is non-zero, $\|E_\omega\pi_\omega(x)E_\omega\xi\|^2 = u(x,x)\|E_\omega\pi_\omega(x^*)E_\omega\xi\|^2>0$ for some $\xi\in\mathcal{H}_\omega$. It follows that $u(x,x)=1$ and $\pi_\omega(x)^{\mathfrak{S}}$ is necessarily a normal operator on $\mathcal{H}_\omega^{\mathfrak{S}}$. In particular,

- $\operatorname{ran}\left(\pi_\omega(x)^{\mathfrak{G}}\right) = \operatorname{ran}\left(\pi_\omega(x^*)^{\mathfrak{G}}\right)$
- $\ker\left(\pi_\omega(x)^{\mathfrak{G}}\right) = \ker\left(\pi_\omega(x^*)^{\mathfrak{G}}\right) = \operatorname{ran}\left(\pi_\omega(x)^{\mathfrak{G}}\right)^{\perp} = \operatorname{ran}\left(\pi_\omega(x^*)^{\mathfrak{G}}\right)^{\perp}$

and the proof is accomplished. □

Thanks to Theorem III.7.1, we can give a necessary and sufficient condition for a symmetric state $\omega \in \mathcal{S}_{\mathfrak{G}}(\mathfrak{A})$ to be $\mathfrak{G}$-abelian, a crucial property for the upcoming De Finetti theorems. For $\omega \in \mathcal{S}_{\mathfrak{G}}(\mathfrak{A})$, let $\operatorname{spt}(\omega) := [\mathfrak{A}_\sigma \colon \omega|_{\mathfrak{A}_\sigma} \neq 0] = [\mathfrak{A}_\sigma \colon \pi_\omega(\mathfrak{A}_\sigma)\xi_\omega \not\subset \mathbb{C}\xi_\omega]$. Also, let $\Delta_+^* := \Delta_+ \backslash \{\iota\} \subset \widehat{G}$. If $\sigma \in \Delta_+^*$, let $T_\sigma := \Delta_+^* \setminus \{\sigma\}^{\perp_u} = \{\tau \in \Delta_+^* \colon u(\sigma, \tau) \neq 1\}$. Then,

- T_σ is a (possibly, empty) symmetric subset of Δ_+^*
- $T_\sigma = T_{\sigma^{-1}}$
- $\tau \in T_\sigma$ if and only if $\sigma \in T_\tau$

In particular, $\pi_\omega\left(\bigoplus_{\tau \in T_\sigma} \mathfrak{A}_\tau\right)^{\mathfrak{G}}$ is a *-closed operator space in $\mathcal{B}(\mathcal{H}_\omega^{\mathfrak{G}})$. We are now ready for the investigation of the ergodic properties of $\mathcal{S}_{\mathfrak{G}}(\mathfrak{A})$. It will turn out that every symmetric state satisfies an invariance property, in general weaker than the G-invariance one.

Corollary III.7.2 (Ergodic properties of symmetric states)
Let $\omega \in \mathcal{S}_{\mathfrak{G}}(\mathfrak{A})$. Then,

(i) $\operatorname{spt}(\omega) \subseteq [\mathfrak{A}_\sigma \colon \pi_\omega(\mathfrak{A}_\sigma)^{\mathfrak{G}} \neq \{0\}] \subseteq [\mathfrak{A}_\sigma \colon \sigma \in \Delta_+]$.
In particular, ω is $\Delta_+^{\perp}$-invariant, where $\Delta_+^{\perp} := \{g \in G \colon \sigma(g) = 1, \sigma \in \Delta_+\}$ is the annihilator of Δ_+.

(ii) ω is *asymptotically abelian in average* if and only if for each $\sigma \in \Delta_+^*$, $\tau \in \widehat{G}$ s.t. $u(\sigma, \tau) \neq 1$,

$$\pi_\omega(\mathfrak{A}_\sigma)^{\mathfrak{G}}\xi_\omega \in \left(\bigcap_{\substack{x \in \mathfrak{A}_\tau \\ b \in \mathfrak{A}}} \ker(\pi_\omega(xb))\right) \cap \mathcal{H}_\omega^{\mathfrak{G}} \tag{III.12}$$

(iii) ω is $\mathfrak{G}$-*abelian* if and only if for each $\sigma \in \Delta_+^*$,

$$\overline{\pi_\omega\left(\mathfrak{A}_\sigma\right)^{\mathfrak{G}}\mathcal{H}_\omega^{\mathfrak{G}}} \perp \overline{\pi_\omega\left(\bigoplus_{\tau \in T_\sigma} \mathfrak{A}_\tau\right)^{\mathfrak{G}}\mathcal{H}_\omega^{\mathfrak{G}}} \tag{III.13}$$

Proof.
Given a homogeneous $x \in \operatorname{spt}(\omega)$, we have $0 < |\omega(x)| \leq \|E_\omega \pi_\omega(x) E_\omega\|$. Therefore, by Theorem III.7.1, $u(x, x) = 1$ i.e. $x \in [\mathfrak{A}_\sigma \colon \sigma \in \Delta_+]$ and point (i) is accomplished. As concerns (ii), by point (iii) of Theorem III.7.1, if $x, y, a, b \in \mathfrak{A}$ are homogeneous and $u(x, y) \neq 1$ (where we can suppose $\partial y \in \Delta_+^*$, taking advantage of point (i)), then $\mathcal{M}\{\omega(a[x, \rho(y)]b)\} = 0$ if and only if $\langle \pi_\omega(xb)E_\omega\pi_\omega(y)\xi_\omega, \pi_\omega(a^*)\xi_\omega\rangle = 0$. By density of $\dotplus_{\sigma \in \widehat{G}} \mathfrak{A}_\sigma$ in $\mathfrak{A}$ and cyclicity of ξ_ω, $\pi_\omega(xb)\pi_\omega(y)^{\mathfrak{G}}\xi_\omega = 0$ i.e. $\pi_\omega(y)^{\mathfrak{G}}\xi_\omega \in \ker(\pi_\omega(xb))$. Lastly, point (i) along with an easy application of point (ii) in Theorem III.7.1, implies that for homogeneous $x, y \in \mathfrak{A}$, $\left[\pi_\omega(x)^{\mathfrak{G}}, \pi_\omega(y)^{\mathfrak{G}}\right] = 0$ whenever $\partial x \in \Delta_-$, $\partial y \in \Delta_-$ or $u(x, y) = 1$. That allows us to reduce to the case where $\partial x \in \Delta_+^*$ and $\partial y \in T_{\partial x}$. In this situation, again by point (ii) in Theorem III.7.1, $\left[\pi_\omega(x)^{\mathfrak{G}}, \pi_\omega(y)^{\mathfrak{G}}\right] = 0$ if and only if $\pi_\omega(x)^{\mathfrak{G}}\pi_\omega(y)^{\mathfrak{G}} = 0$, that is

$$\langle \pi_\omega(x^*)^{\mathfrak{G}}\xi, \pi_\omega(y)^{\mathfrak{G}}\eta\rangle_{\mathcal{H}_\omega^{\mathfrak{G}}} = 0, \quad \xi, \eta \in \mathcal{H}_\omega^{\mathfrak{G}}$$

or $\operatorname{ran}\left(\pi_\omega(x)^{\mathfrak{G}}\right) = \operatorname{ran}\left(\pi_\omega(x^*)^{\mathfrak{G}}\right) \perp \operatorname{ran}\left(\pi_\omega(y)^{\mathfrak{G}}\right)$, whence Equation III.13 follows. □

Remark III.7.3
$\Delta_+^\perp$ is evidently a closed subgroup of G, hence compact. Point (i) in Corollary III.7.2 tells us that $\mathcal{S}_{\mathfrak{G}}(\mathfrak{A}) \subseteq \mathcal{S}_{\Delta_+^\perp}(\mathfrak{A})$. Observe that $\mathcal{S}_{\Delta_+^\perp}(\mathfrak{A})$ is a weakly-* compact and convex set, which contains $\mathcal{S}_G(\mathfrak{A})$.

Apparently, Corollary III.7.2 raises issues on the feasibility of the ergodic analysis of symmetric states on a twisted chain. Precisely:

(1) when u is *symplectic*, i.e. $\Delta_+ = \widehat{G}$, then $\Delta_+^\perp = (0)$ and point (i) of Corollary III.7.2 does not say much about the group action invariance of ω. This is not good news, since every non-degenerate $u \in \mathsf{A}(\widehat{G})$ is symplectic whenever, for instance, either $\widehat{G}$ has finite *odd* order, or $\widehat{G} = \mathbb{Z}^n$, $n \geq 2$.

(2) asymptotic abelianness in average might be hard to achieve from Equation III.12, when $\pi_\omega(\mathfrak{A}_\sigma)^{\mathfrak{G}}\xi_\omega \neq \{0\}$ for some $\sigma \in \Delta_+^*$.

(3) even Equation III.13 in point (iii) is not easy to apply in general, since the explicit form of the GNS covariant representation $(\mathcal{H}_\omega, \pi_\omega, U_\omega, \xi_\omega)$ of a symmetric state $\omega \in \mathcal{S}_{\mathfrak{G}}(\mathfrak{A})$ is, to the best of our knowledge, unclear, let alone the $U_\omega(\mathfrak{G})$-invariant Hilbert space $\mathcal{H}_\omega^{\mathfrak{G}}$. We only observe that if $(\widehat{G}, u) \sim (\mathbb{Z}_p^2, w_p)$ for prime $p \geq 3$ (see Section III.5), one can prove that for each $\sigma \in \mathbb{Z}_p^2 \setminus \{0\}$, $\langle\sigma\rangle^* := \langle\sigma\rangle \setminus \{0\} = \{\tau \in \mathbb{Z}_p^2 \setminus \{0\} \colon u(\sigma, \tau) = 1\}$ and Equation III.13 translates into

$$\overline{\pi_\omega\left(\bigoplus_{\tau \in \langle\sigma\rangle^*} \mathfrak{A}_\tau\right)^{\mathfrak{G}}(\mathcal{H}_\omega^{\mathfrak{G}})} \perp \overline{\pi_\omega\left(\bigoplus_{\tau \in T_\sigma} \mathfrak{A}_\tau\right)^{\mathfrak{G}}(\mathcal{H}_\omega^{\mathfrak{G}})}, \quad \sigma \in \Delta_+^*.$$

Nonetheless, we can at least fully overcome question (3) by requiring the non-degenerate bicharacter $u \in \mathsf{A}(\widehat{G})$ to satisfy $u|_{\Delta_+} \equiv 1$ (in general, $u|_{\Delta_+} \in \mathsf{A}(\Delta_+)$ is just alternating). This requirement, though resulting to be significantly restrictive, provides us a new model, never addressed before, upon which the ergodic theory of symmetric states can well be performed: the *Klein twisted chain*. The following simple algebraic result in group theory explains why. We take the occasion to express our gratitude to the anonymous user of *Mathematics Stack Exchange* who gave elucidations on this result (see [111]).

Proposition III.7.4
Let $\mathcal{G}$ be a discrete abelian group and consider a non-degenerate, skew-symmetric bicharacter $u \in \mathsf{A}(\mathcal{G})$ s.t. $u|_{\Delta_+} \equiv 1$. Then, one of the following occurs:

(1) $\mathcal{G} = (0)$, $u \equiv 1$ and $\Delta_+ = (0)$ (*trivial* bicharacter)

(2) $\mathcal{G} \cong \mathbb{Z}_2$, $u \sim u_{\mathtt{F}}$ and $\Delta_+ = (0)$ (*Fermi* bicharacter)

(3) $\mathcal{G} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, $u \sim u_{\mathtt{K}}$ and $\Delta_+ \cong \mathbb{Z}_2$ (*Klein* bicharacter)

Proof.
Recall that $|\mathcal{G} \colon \Delta_+| \leq 2$. If $|\mathcal{G} \colon \Delta_+| = 1$ (i.e. $\Delta_+ = \mathcal{G}$), then $u \equiv 1$, thus it is non-degenerate if and only if $\mathcal{G} = (0)$. Therefore, from now on, suppose $|\mathcal{G} \colon \Delta_+| = 2$. The inclusion map $\iota \colon \Delta_+ \hookrightarrow \mathcal{G}$ induces a canonical surjection $\pi \colon \widehat{\mathcal{G}} \twoheadrightarrow \widehat{\Delta_+}$ s.t. $|\ker\pi| = \left|\widehat{\mathcal{G}/\Delta_+}\right| = |\mathbb{Z}_2| = 2$. Since u is non-degenerate, the map

$$\begin{aligned}\gamma \colon \mathcal{G} &\hookrightarrow \widehat{\mathcal{G}}\\ g &\mapsto u(g, \cdot)\end{aligned}$$

is a group monomorphism. On the other hand, since $u|_{\Delta_+} \equiv 1$, the composition

$$\Delta_+ \overset{\gamma\circ\iota}{\hookrightarrow} \widehat{\mathcal{G}} \overset{\pi}{\twoheadrightarrow} \widehat{\Delta_+}$$

is the trivial homomorphism. This means that $(\gamma \circ \iota)(\Delta_+) \subset \ker \pi$ and hence

$$|\Delta_+| = |(\gamma \circ \iota)(\Delta_+)| \leq |\ker \pi| = 2\,.$$

Since $|\mathcal{G} \colon \Delta_+| = 2$, this forces $\mathcal{G}$ to have order $2 \leq |\mathcal{G}| \leq 4$. More correctly, since $|\mathcal{G}| = |\mathcal{G} \colon \Delta_+||\Delta_+| = 2|\Delta_+|$, $|\mathcal{G}| = 2, 4$. If $|\mathcal{G}| = 2$, then $\mathcal{G} \cong \mathbb{Z}_2$ and there exists a unique non-degenerate bicharacter $u \in \mathsf{B}(\mathcal{G})$. Precisely, $u \sim u_{\mathtt{F}}$, thus u is skew-symmetric (equivalently, symmetric) and $\Delta_+ = (0)$. If $|\mathcal{G}| = 4$, then either $\mathcal{G} \cong \mathbb{Z}_4$ or $\mathcal{G} \cong K_4 = \mathbb{Z}_2 \times \mathbb{Z}_2$. On the one hand, the only two non-degenerate bicharacters on $\mathbb{Z}_4$ have the form

$$u_\varepsilon(\sigma, \tau) = i^{\varepsilon\sigma\tau} \quad (\sigma, \tau \in \mathbb{Z}_4)$$

for $\varepsilon \in \{\pm 1\}$, thus they are symmetric, but not skew-symmetric. On the other hand, K_4 admits a unique (up to equivalence) non-degenerate skew-symmetric bicharacter s.t. $|K_4 \colon \Delta_+| = 2$, that is $u_{\mathtt{K}} \in \mathsf{A}(K_4)$. Observe that $\Delta_+ = \{(0,0), (1,1)\}$ and $u_{\mathtt{K}}|_{\Delta_+ \times \Delta_+} \equiv 1$. Therefore, $\mathcal{G} \cong K_4 = \mathbb{Z}_2 \times \mathbb{Z}_2$ and $u \sim u_{\mathtt{K}}$. □

As already mentioned, by point (iii) in Corollary III.7.2, if $\widehat{G}$ (and, consequently, the group G acting on the tensor factor algebra $\mathfrak{B}$) is one of the three ones resulting from Proposition III.7.4, any symmetric state $\omega \in \mathcal{S}_{\mathfrak{S}}(\mathfrak{A})$ must be $\mathfrak{S}$-abelian. The first two cases were thoroughly studied by Størmer in [76] and Fidaleo in [31], respectively. We briefly report their results in the following two subsections. For starters, we define the *Størmer's shuffles* as the sequence $(\rho_n)_{n\geq 1} \subset \mathfrak{S}$ s.t.

$$\rho_n(j) := \begin{cases} j + 2^{n-1} \text{ if } j \in \mathbf{2}^{n-1} = \{1, \dots, 2^{n-1}\} \\ j - 2^{n-1} \text{ if } j \in \mathbf{2}^n \setminus \mathbf{2}^{n-1} = \{2^{n-1}+1, \dots, 2^n\} \\ j \text{ if } j \geq 2^n + 1 \end{cases} \tag{III.14}$$

for each $n \geq 1$. For completeness, notice that $\rho_1 = (1\ 2)$ and that $\rho_n \in \mathcal{A}_{\mathbf{2}^n}$ for every $n \geq 2$, since the number of inversions of ρ_n is $\mathsf{inv}(\rho_n) = 2^{2n-2}$ $(n \geq 1)$.

III.7.1 $G = \widehat{G} = (0)$, $ⓐ = \otimes$: the trivial twisted C^*-chain

This is the case where no twists are involved, and $\mathfrak{A}$ corresponds to the infinite (minimal) C^*-tensor product $\mathfrak{A} := \overline{\underset{n\in\mathbb{N}}{\otimes} \mathfrak{B}}^{\min}$ analyzed by Guichardet in Sections 2.3, 2.4, 2.5 and 2.6 of [43] (p. 18-28). The following result combines Lemma 2.1 in [76] (p. 52), Theorem III.2.8 and point (4) in Theorem 3.1 of [77] (p. 9). Observe that $\mathfrak{A}$ is abelian, simple, nuclear, or separable if and only if $\mathfrak{B}$ is (since minimal tensor products and C^*-inductive limits preserve these properties, see for instance [78]).

Theorem III.7.5
Let $\mathfrak{B}$ be a unital C^*-algebra and $\mathfrak{A}$ its minimal C^*-chain, upon which $\mathfrak{S}$ acts canonically by permutations of the indices. Then, $\lim\limits_{n\to+\infty} \|[\rho_n(a), b]\| = 0$ for every $a, b \in \mathfrak{A}$. Consequently, the C^*-system $(\mathfrak{A}, \mathfrak{S})$ is asymptotically abelian and $\mathfrak{S}$ acts largely on $\mathfrak{A}$. In particular, the family of symmetric states $\mathcal{S}_{\mathfrak{S}}(\mathfrak{A})$ is a Choquet simplex.

The largeness of the $\mathfrak{S}$-action on $\mathfrak{A}$ also guarantees the existence and uniqueness of a σ-weakly continuous, $\mathfrak{S}$-invariant expectation $\Phi_\omega\colon \pi_\omega(\mathfrak{A})'' \twoheadrightarrow \mathfrak{Z}_\omega \cap U_\omega(\mathfrak{S})'$ for any $\omega \in \mathcal{S}_\mathfrak{S}(\mathfrak{A})$, where $\mathfrak{Z}_\omega := \pi_\omega(\mathfrak{A})' \cap \pi_\omega(\mathfrak{A})''$ is the center of the GNS von Neumann algebra $\pi_\omega(\mathfrak{A})''$ (see Theorem 3.1 in [77], p. 9). Moreover, $(\Phi_\omega \circ \pi_\omega)(a) = w\text{-}\lim_{n\to+\infty}(\pi_\omega \circ \rho_n)(a)$, $a \in \mathfrak{A}$ (see Lemma 2.6 in [76], p. 56).

To complete the ergodic description of $\mathcal{S}_\mathfrak{S}(\mathfrak{A})$, Størmer also characterizes the elements in $\mathcal{E}_\mathfrak{S}(\mathfrak{A})$ (i.e. the ergodic symmetric states of $\mathfrak{A}$) as the ones of the form $\prod_{n\in\mathbb{N}}\psi$, with $\psi \in \mathcal{S}(\mathfrak{B})$ any fixed state of $\mathfrak{B}$. Precisely, Theorems 2.7 (p. 57) and 2.8 (p. 58) in [76], Theorem III.2.3 and Theorem III.2.7 give the following remarkable unified result.

Theorem III.7.6 (Characterization of $\mathcal{E}_\mathfrak{S}(\mathfrak{A})$, $\mathfrak{A}$ minimal C^*-chain)
Let $\omega \in \mathcal{S}_\mathfrak{S}(\mathfrak{A})$. Then, the following are equivalent:

(i) $\omega \in \mathcal{E}_\mathfrak{S}(\mathfrak{A})$

(ii) ω is *strongly clustering*: $\lim_{n\to+\infty} \omega(\rho_n(a)b) = \omega(a)\omega(b)$, $a, b \in \mathfrak{A}$

(iii) ω is *weakly clustering*: $\inf_{x\in\mathrm{co}(G\cdot a)} |\omega(xb) - \omega(a)\omega(b)| = 0$, $a, b \in \mathfrak{A}$

(iv) there exists $\psi \in \mathcal{S}(\mathfrak{B})$ s.t. $\omega = \prod_{n\in\mathbb{N}}\psi$

In particular, $\mathcal{E}_\mathfrak{S}(\mathfrak{A}) = \left\{\prod_{n\in\mathbb{N}}\psi\right\}_{\psi\in\mathcal{S}(\mathfrak{B})}$ is weakly-* closed, thus making $\mathcal{S}_\mathfrak{S}(\mathfrak{A})$ a Bauer simplex in $\mathcal{S}(\mathfrak{A})$. Precisely, $(\mathcal{S}(\mathfrak{B}), \tau_{w^*})$ and $(\mathcal{E}_\mathfrak{S}(\mathfrak{A}), \tau_{w^*})$ are homeomorphic via the mapping

$$\begin{aligned}\iota\colon \mathcal{S}(\mathfrak{B}) &\to \mathcal{E}_\mathfrak{S}(\mathfrak{A})\\ \psi &\to \prod_{n\in\mathbb{N}}\psi\end{aligned}$$

Lastly, there exists a $\mathfrak{S}$-invariant, densely ranged p.u. map of C^*-algebras

$$T\colon \mathfrak{A} \to \mathcal{C}(\mathcal{E}_\mathfrak{S}(\mathfrak{A}))$$

s.t. its transpose $T^{\mathrm{t}}\colon \mathcal{M}_1(\mathcal{E}_\mathfrak{S}(\mathfrak{A})) \to \mathcal{S}_\mathfrak{S}(\mathfrak{A})$ is an affine homeomorphism.

We can then conclude the present subsection by recovering the following version of De Finetti theorem for minimal C^*-chains.

Theorem III.7.7 (De Finetti theorem for minimal C^*-chains)
Let $(\mathfrak{A}, \mathfrak{S})$ be the C^*-system associated to a unital C^*-algebra $\mathfrak{B}$, as in Theorem III.7.5. Then, for each $\varphi \in \mathcal{S}_\mathfrak{S}(\mathfrak{A})$, there exists a unique $\prec$-maximal $\mu_\varphi \in \mathcal{M}_1(\mathcal{S}_\mathfrak{S}(\mathfrak{A}))$ s.t.

$$\varphi(a) = \int_{\mathcal{S}_\mathfrak{S}(\mathfrak{A})} \omega(a)\,\mathrm{d}\mu_\varphi(\omega), \qquad a \in \mathfrak{A}. \tag{III.15}$$

In particular, μ_φ is pseudo-supported by $\mathcal{E}_\mathfrak{S}(\mathfrak{A}) = \left\{\prod_{n\in\mathbb{N}}\psi\right\}_{\psi\in\mathcal{S}(\mathfrak{B})}$ i.e. $\mu_\varphi(B) = 1$ for every $B \in \mathcal{B}_0(\mathcal{S}_\mathfrak{S}(\mathfrak{A}))$ containing $\mathcal{E}_\mathfrak{S}(\mathfrak{A})$. The relative weak-* topology on the unit ball $B_{\mathfrak{A}^*}$ of $\mathfrak{A}^*$ is metrizable if and only if $\mathfrak{B}$ is separable (equivalently, countably generated), in which case μ_φ is *supported* by $\mathcal{E}_\mathfrak{S}(\mathfrak{A})$ and Equation III.15 becomes

$$\varphi(a) = \int_{\mathcal{E}_\mathfrak{S}(\mathfrak{A})} \omega(a)\,\mathrm{d}\mu_\varphi(\omega), \qquad a \in \mathfrak{A}. \tag{III.16}$$

Proof.
This is a restatement of Theorem III.7.5, Theorem III.2.3, Proposition III.2.4 and Theorem III.7.6. Notice that the relative weak-* topology on the unit ball $B_{\mathfrak{A}^*}$ is metrizable if and only if $\mathfrak{A}$ is separable (see Theorem 2.6.23 in [95], p. 231), or equivalently $\mathfrak{B}$ is. □

III.7.2 $G = \widehat{G} = \mathbb{Z}_2$, $\textcircled{u} = \textcircled{F}$: the Fermi twisted C^*-chain

This case was firstly studied by Crismale and Fidaleo in [18] in the particular situation where $\mathfrak{B}_n = \mathfrak{B} := M_2(\mathbb{C})$ for every $n \in \mathbb{N}$ and the inner action $\mathbb{Z}_2 \overset{\beta}{\curvearrowright} M_2(\mathbb{C})$ is defined by $\beta_x := \mathrm{ad}_{\left[\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right]^x}$, $x \in \mathbb{Z}_2 = \{0,1\}$, so that $\mathfrak{B} = M_2(\mathbb{C})_+ \oplus M_2(\mathbb{C})_- = \mathrm{span}_{\mathbb{C}}\{E_{11}, E_{22}\} \oplus \mathrm{span}_{\mathbb{C}}\{E_{12}, E_{21}\}$. Once defined the *CAR* (*Canonical Anticommutation Relations*) *algebra* as the (unital) universal C^*-algebra $\mathrm{CAR}(\mathbb{N}) := C^*\left(a_j, a_j^\dagger \,\middle|\, a_j^* = a_j^\dagger, \{a_j, a_k\} = \{a_j^\dagger, a_k^\dagger\} = 0, \{a_j^\dagger, a_k\} = \delta_{jk}\mathbb{1},\ j,k \in \mathbb{N}\right)$ and the *Fermi twisted* C^**-chain* of $\mathfrak{B}$ as $\mathfrak{A} := \textcircled{F}_{n\in\mathbb{N}} \mathfrak{B}$ (where $\textcircled{F}$ is the twisted tensor structure induced by the unique non-degenerate bicharacter of $\widehat{\mathbb{Z}_2} \cong \mathbb{Z}_2$, $u_{\mathtt{F}} \in \mathsf{A}(\mathbb{Z}_2)$), the map defined on the matrix unit at the n-th site ($n \in \mathbb{N}$) by

$$\begin{aligned}
\phi\colon \mathfrak{A} &\to \mathrm{CAR}(\mathbb{N})\\
E_{11}^{(n)} &\mapsto a_n a_n^\dagger\\
E_{12}^{(n)} &\mapsto a_n\\
E_{21}^{(n)} &\mapsto a_n^\dagger\\
E_{22}^{(n)} &\mapsto a_n^\dagger a_n
\end{aligned}$$

is a C^*-algebra isomorphism (here, we have identified $E_{ij}^{(n)}$ with its image under the canonical embedding $\psi_n\colon \mathfrak{B}_n \hookrightarrow \textcircled{F}_{n\in\mathbb{N}} \mathfrak{B}$). The analysis was then generalized by Fidaleo in [31] to a Fermi C^*-chain of a general unital C^*-algebra $\mathbb{Z}_2$-graded $\mathfrak{B}$. The interesting outcome of this analysis is the following: every symmetric state $\omega \in \mathcal{S}_{\mathfrak{S}}(\mathfrak{A})$ on a Fermi C^*-chain $\mathfrak{A}$ is necessarily *even*, that is invariant under the action $\mathbb{Z}_2 \overset{\delta^{(\beta)}}{\curvearrowright} \mathfrak{A}$ (or equivalently $\omega \circ \Theta = \omega$, where $\Theta := (\delta^{(\beta)})_1 = \textcircled{F}_{n\in\mathbb{N}} \beta_1 = \textcircled{F}_{n\in\mathbb{N}} \vartheta \in \mathsf{Aut}(\mathfrak{A})$, $\vartheta \in \mathsf{Aut}(\mathfrak{B})$ being the involutive automorphism realizing the $\mathbb{Z}_2$-grading on $\mathfrak{B}$). In other words, $\mathcal{S}_{\mathfrak{S}}(\mathfrak{A}) \subset \mathcal{S}_+(\mathfrak{A}) \cong \mathcal{S}(\mathfrak{A}_+)$. The following result combines Theorem 5.1 (p. 17) and Propositions 5.4 (p. 19) in [31].

Theorem III.7.8
Let $(\mathfrak{B}, \mathbb{Z}_2, \beta)$ be a C^*-system and $\mathfrak{A}$ its Fermi C^*-chain, upon which $\mathfrak{S}$ acts by permutations of the indices as exposed in Section III.6. Then, each $\omega \in \mathcal{S}_{\mathfrak{S}}(\mathfrak{A})$ is

(i) even: $\omega \circ \delta^{(\beta)} = \omega$

(ii) asymptotically abelian in average: $\mathcal{M}\{\omega(c[\rho(a), b]d)\} = 0$ for every $a, b, c, d \in \mathfrak{A}$

Consequently, $\mathfrak{S}$ acts largely on $\mathfrak{A}$ and $\mathcal{S}_{\mathfrak{S}}(\mathfrak{A})$ is a Choquet simplex.

Again, the largeness of the $\mathfrak{S}$-action gives a unique normal, $\mathfrak{S}$-invariant expectation $\Phi_\omega\colon \pi_\omega(\mathfrak{A})'' \twoheadrightarrow \mathfrak{Z}_\omega \cap U_\omega(\mathfrak{S})'$ for each symmetric state $\omega \in \mathcal{S}_{\mathfrak{S}}(\mathfrak{A})$, satisfying

$$(\Phi_\omega \circ \pi_\omega)(a) = \mathcal{M}\{(\pi_\omega \circ \rho)(a)\}, \quad a \in \mathfrak{A}$$

in the weak operator topology of $\mathcal{B}(\mathcal{H}_\omega)$ (see Proposition 5.5, p. 19, in [31]).
It is really worth noticing that the asymptotic abelianness property of $(\mathfrak{A}, \mathfrak{S})$, as defined by Størmer in [77], p. 17 (see Section III.2), abruptly fails to hold in general, in the Fermi

twisted case. For instance, if $\mathfrak{B}$ is a non-abelian $\mathbb{Z}_2$-graded C^*-algebra and $a, b \in \mathfrak{B}_-$ (with a selfadjoint), then for $\rho \in \mathfrak{S}$

$$[\rho(a), b] = \begin{cases} [a,b] \textcircled{F} \mathbb{1}_{\mathfrak{B}} \textcircled{F} \dots \neq 0 & \text{if } 1 \in \mathrm{Fix}(\rho) \\ -2(\mathbb{1}_{\mathfrak{B}} \textcircled{F} \cdots \textcircled{F} \mathbb{1}_{\mathfrak{B}} \textcircled{F} b \textcircled{F} \mathbb{1}_{\mathfrak{B}} \textcircled{F} \cdots \textcircled{F} \mathbb{1}_{\mathfrak{B}} \textcircled{F} a \textcircled{F} \mathbb{1}_{\mathfrak{B}} \textcircled{F} \dots) \neq 0 & \text{if } 1 \notin \mathrm{Fix}(\rho) \end{cases}$$

so that there cannot exist $\{\rho_{n,a}\}_{n\in\mathbb{N}} \subseteq \mathfrak{S}$ s.t $\lim_{n\to+\infty} \|[\rho_{n,a}(a), b]\| = 0$.

This is why in order to accomplish results (i) and (ii) in Theorem III.7.8, the combinatorial Lemma III.6.3 (unnecessary to Størmer in the trivial twisted case, but exploited by us to achieve Theorem III.7.1) is really required here. It will be of primary importance for the investigation of the third case too.

In the spirit of Størmer's work, $\mathcal{E}_{\mathfrak{S}}(\mathfrak{A})$ is fully descripted in the Fermi case too. Let us collect Corollary 5.2 (p. 18) and Theorems 6.1 (p. 21), 6.3 (p. 24) of [31] in the following result.

Theorem III.7.9 (Characterization of $\mathcal{E}_{\mathfrak{S}}(\mathfrak{A})$, $\mathfrak{A}$ Fermi C^*-chain)
Let $\omega \in \mathcal{S}_{\mathfrak{S}}(\mathfrak{A})$. Then, the following are equivalent:

(i) $\omega \in \mathcal{E}_{\mathfrak{S}}(\mathfrak{A})$

(ii) ω is *strongly clustering*: $\lim_{n\to+\infty} \omega(\rho_n(a)b) = \omega(a)\omega(b)$, $a, b \in \mathfrak{A}$

(iii) ω is *weakly clustering in average*: $\mathcal{M}\{\omega(\rho(a)b)\} = \omega(a)\omega(b)$, $a, b \in \mathfrak{A}$

(iv) there exists $\psi \in \mathcal{S}_+(\mathfrak{B}) \cong \mathcal{S}(\mathfrak{B}_+)$ s.t. $\omega = \prod_{n\in\mathbb{N}} \psi$

In particular, $\mathcal{E}_{\mathfrak{S}}(\mathfrak{A}) = \left\{\prod_{n\in\mathbb{N}} \psi\right\}_{\psi\in\mathcal{S}_+(\mathfrak{B})}$ is weakly-* closed, thus making $\mathcal{S}_{\mathfrak{S}}(\mathfrak{A})$ a Bauer simplex in $\mathcal{S}(\mathfrak{A})$. Precisely, $(\mathcal{S}_+(\mathfrak{B}), \tau_{w^*})$ and $(\mathcal{E}_{\mathfrak{S}}(\mathfrak{A}), \tau_{w^*})$ are homeomorphic via the mapping

$$\iota \colon \mathcal{S}_+(\mathfrak{B}) \to \mathcal{E}_{\mathfrak{S}}(\mathfrak{A})$$
$$\varphi \to \prod_{n\in\mathbb{N}} \psi$$

Lastly, there exists a $\mathfrak{S}$-invariant, densely ranged p.u. map of C^*-algebras

$$T \colon \mathfrak{A} \to \mathcal{C}(\mathcal{S}_+(\mathfrak{B}))$$

s.t. its transpose $T^{\mathrm{t}} \colon \mathcal{M}_1(\mathcal{S}_+(\mathfrak{B})) \to \mathcal{S}_{\mathfrak{S}}(\mathfrak{A})$ is an affine homeomorphism.

From Theorem III.7.8, Theorem III.2.3, Proposition III.2.4 and Theorem III.7.9, De Finetti theorem for Fermi twisted C^*-chains now becomes as follows.

Theorem III.7.10 (De Finetti theorem for Fermi twisted C^*-chains)
Let $(\mathfrak{A}, \mathfrak{S})$ be the C^*-system associated to a unital C^*-algebra $\mathbb{Z}_2$-graded $\mathfrak{B}$, as in Theorem III.7.8. Then, for each $\varphi \in \mathcal{S}_{\mathfrak{S}}(\mathfrak{A})$, there exists a unique $\prec$-maximal $\mu_\varphi \in \mathcal{M}_1(\mathcal{S}_{\mathfrak{S}}(\mathfrak{A}))$ s.t.

$$\varphi(a) = \int_{\mathcal{S}_{\mathfrak{S}}(\mathfrak{A})} \omega(a) \, \mathrm{d}\mu_\varphi(\omega), \qquad a \in \mathfrak{A}. \tag{III.17}$$

In particular, μ_φ is pseudo-supported by $\mathcal{E}_{\mathfrak{S}}(\mathfrak{A}) = \left\{\prod_{n\in\mathbb{N}} \psi\right\}_{\psi\in\mathcal{S}_+(\mathfrak{B})}$ i.e. $\mu_\varphi(B) = 1$ for every $B \in \mathcal{B}_0(\mathcal{S}_{\mathfrak{S}}(\mathfrak{A}))$ containing $\mathcal{E}_{\mathfrak{S}}(\mathfrak{A})$. The relative weak-* topology on the unit ball $B_{\mathfrak{A}^*}$ of $\mathfrak{A}^*$ is metrizable if and only if $\mathfrak{B}$ is separable (equivalently, countably generated), in which case μ_φ is *supported* by $\mathcal{E}_{\mathfrak{S}}(\mathfrak{A})$ and Equation III.17 becomes

$$\varphi(a) = \int_{\mathcal{E}_{\mathfrak{S}}(\mathfrak{A})} \omega(a)\,\mathrm{d}\mu_\varphi(\omega), \qquad a \in \mathfrak{A}\,. \tag{III.18}$$

III.8. $G = \widehat{G} = K_4$, ⓤ = Ⓚ: the Klein twisted C^*-chain

Here we are with the last case where a satifying ergodic analysis of $\mathcal{S}_{\mathfrak{S}}(\mathfrak{A})$ can be performed: the *Klein twisted C^*-chain*. Let $K_4 := \mathbb{Z}_2 \times \mathbb{Z}_2$ be the Klein 4-group, acting on a unital C^*-algebra $\mathfrak{B}$. Clearly, $\widehat{K_4} \cong K_4$ and the bicharacter on K_4 $u_{\mathrm{K}}(\mathbf{x}, \mathbf{y}) := (-1)^{\mathbf{x}^t \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right] \mathbf{y}}$ $(\mathbf{x}, \mathbf{y} \in K_4)$ introduced in Equation III.7 is non-degenerate, skew-symmetric and non-symplectic. Moreover, $\Delta_+ = \langle (1,1) \rangle \cong \mathbb{Z}_2$ and $u|_{\Delta_+} \equiv 1$. Since the annihilator $\Delta_+^\perp$ of Δ_+ is again $\{(0,0),(1,1)\} = \langle (1,1) \rangle$, we can identify it with Δ_+ itself, even if the latter formally lies in the acting group K_4, while the former in its isomorphic dual $\widehat{K_4}$. Since we are going to deal with this situation only for the whole present section, this choice of notation should not confuse the reader. We start with a result analogous to Theorem III.7.5 in the case $G = (0)$ and to Theorem III.7.8 in the case $G = \mathbb{Z}_2$.

Theorem III.8.1
Let $(\mathfrak{B}, K_4, \beta)$ be a C^*-system and $\mathfrak{A}$ its Klein C^*-chain, upon which $\mathfrak{S}$ acts by permutations of the indices as exposed in Section III.6. Then, each $\omega \in \mathcal{S}_{\mathfrak{S}}(\mathfrak{A})$ is

(i) $\langle (1,1) \rangle$-invariant: $\omega \circ \delta^{(\beta|\Delta_+)} = \omega$. In other words, $\omega \in \mathcal{S}_{\Delta_+}(\mathfrak{A}) \cong \mathcal{S}(\mathfrak{A}_{(0,0)} \oplus \mathfrak{A}_{(1,1)})$.

(ii) asymptotically abelian in average if and only if

$$\pi_\omega(\mathfrak{A}_{(1,1)})^{\mathfrak{S}} \xi_\omega \in \ker(\pi_\omega(xb)), \ker(\pi_\omega(yb))$$

for every $b \in \mathfrak{A}$, $x \in \mathfrak{A}_{(1,0)}$, $y \in \mathfrak{A}_{(0,1)}$.

(iii) $\mathfrak{S}$-abelian

Consequently, $\mathcal{S}_{\mathfrak{S}}(\mathfrak{A})$ is a Choquet simplex.

Proof.
These are simple restatements of points (i), (ii) and (iii) in Corollary III.7.2. For point (iii), observe that $\sigma \in \Delta_+^*$ if and only if $\sigma = (1,1)$, in which case $T_\sigma = \varnothing$, thence the necessary and sufficient condition Equation III.13 for $\mathfrak{S}$-abelianness of ω is trivially satisfied. Therefore, every $\omega \in \mathcal{S}_{\mathfrak{S}}(\mathfrak{A})$ is $\mathfrak{S}$-abelian, or equivalently $\mathcal{S}_{\mathfrak{S}}(\mathfrak{A})$ is a Choquet simplex, by Theorem III.2.5. □

Theorem III.8.1 does not guarantee largeness of the $\mathfrak{S}$-action. Anyway, we can still try to recover a full description of $\mathcal{E}_{\mathfrak{S}}(\mathfrak{A})$. To accomplish that, we firstly need to show that the product functional of two Δ_+-invariant states on a C^*-system of the form $(\mathfrak{B}, K_4, \beta)$ is always algebraically positive on the involutive algebra $\mathfrak{B}$ Ⓚ $\mathfrak{B}$, hence providing a well-defined GNS representation of $\mathfrak{B}$ Ⓚ $\mathfrak{B}$ consisting of bounded $\mathbb{C}$-linear operators on a Hilbert space. Before this result, we would like to point out that if $\omega \in \mathcal{S}_{\Delta_+}(\mathfrak{B})$, then its GNS Δ_+-covariant representation $(\mathcal{H}_\omega, \pi_\omega, \xi_\omega, U_\omega)$ of $\mathfrak{B}$ induces a K_4-covariant representation $(\ell^2(\mathbb{Z}_2) \otimes \mathcal{H}_\omega, \Pi, V)$ of $\mathfrak{B}$ by the formulas

$$\begin{cases} \Pi(b)(\delta_x \otimes \xi) := \delta_x \otimes \pi_\omega\left(\beta^x_{(1,0)}(b)\right)\xi, & x \in \mathbb{Z}_2, b \in \mathfrak{B}, \xi \in \mathcal{H}_\omega \\ V_{(x,y)} := \Sigma_{\mathbb{C}^2}^{x+y} \otimes U_\omega^y, & x, y \in \mathbb{Z}_2 \end{cases}$$

where $\Sigma_{\mathbb{C}^2} \in \mathcal{U}(\mathbb{C}^2)$ is the swapping unitary operator of $\mathbb{C}^2$, implemented by the Pauli matrix $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ w.r.t. the canonical basis $\{e_1, e_2\}$. Evidently, Π is faithful iff π_ω (equivalently, $\pi_{\omega \circ \beta_{(1,0)}}$) is.

Lemma III.8.2
Let $(\mathfrak{B}, K_4, \beta)$ be a C^*-system and consider $u_{\mathbb{K}} \in \mathsf{A}(K_4)$. If $\omega, \varphi \in \mathcal{S}(\mathfrak{B})$ and $\mathrm{spt}(\omega), \mathrm{spt}(\varphi) \subset \mathfrak{B}_{(0,0)} \oplus \mathfrak{B}_{(1,1)}$ (equivalently, $\omega, \varphi \in \mathcal{S}_{\Delta_+}(\mathfrak{B})$), then $\omega \times \varphi$ is a state on $\mathfrak{B} \circledast \mathfrak{B}$ and

$$|(\omega \times \varphi)(x)| \leq (\omega \times \varphi)(x^* x)^{1/2}, \quad x \in \mathfrak{B} \circledast \mathfrak{B}\,.$$

In particular, $\pi_{\omega\times\varphi}$ uniquely extends to a representation of $\mathfrak{B} \circledast \mathfrak{B}$ acting by bounded operators on the Hilbert space $\mathcal{H}_{\omega\times\varphi}$.

Proof.
Let $x := \sum_{i=1}^{n} a^{(i)} \odot b^{(i)} = \sum_{i=1}^{n} \left(a^{(i)}_{\Delta_+} + a^{(i)}_{\Delta_-}\right) \odot \left(b^{(i)}_{\Delta_+} + b^{(i)}_{\Delta_-}\right) \in \mathfrak{B} \circledast \mathfrak{B}$. Then,

$$x^* x = \sum_{i,j=1}^{n} \left[\left(a^{(i)}_{\Delta_+} + a^{(i)}_{\Delta_-}\right) \odot \left(b^{(i)}_{\Delta_+} + b^{(i)}_{\Delta_-}\right)\right]^* \left[\left(a^{(j)}_{\Delta_+} + a^{(j)}_{\Delta_-}\right) \odot \left(b^{(j)}_{\Delta_+} + b^{(j)}_{\Delta_-}\right)\right]$$

where for every $i = 1, \ldots, n$

$$\left[\left(a^{(i)}_{\Delta_+} + a^{(i)}_{\Delta_-}\right) \odot \left(b^{(i)}_{\Delta_+} + b^{(i)}_{\Delta_-}\right)\right]^* = \left(a^{(i)}_{\Delta_+} \odot b^{(i)}_{\Delta_+}\right)^\dagger + \sum_{g\in\Delta_-} \left(\beta_g(a^{(i)}) \odot b^{(i)}_g\right)^\dagger + \sum_{g\in\Delta_+} \left(\beta_g(a^{(i)}_{\Delta_-}) \odot b^{(i)}_g\right)^\dagger .$$

Hence, for each $i, j = 1, \ldots, n$,

$$\begin{aligned}
&\left[\left(a^{(i)}_{\Delta_+} + a^{(i)}_{\Delta_-}\right) \odot \left(b^{(i)}_{\Delta_+} + b^{(i)}_{\Delta_-}\right)\right]^* \left[\left(a^{(j)}_{\Delta_+} + a^{(j)}_{\Delta_-}\right) \odot \left(b^{(j)}_{\Delta_+} + b^{(j)}_{\Delta_-}\right)\right] = \\
&\qquad = (a^{(i)}_{\Delta_+} \odot b^{(i)}_{\Delta_+})^\dagger \cdot (a^{(j)}_{\Delta_+} \odot b^{(j)}) + \sum_{g\in\Delta_-} (\beta_g(a^{(i)}) \odot b^{(i)}_g)^\dagger \cdot (\beta_g(a^{(j)}_{\Delta_+}) \odot b^{(j)}) + \\
&\quad + \sum_{g\in\Delta_+} (\beta_g(a^{(i)}_{\Delta_-}) \odot b^{(i)}_g)^\dagger \cdot (a^{(j)}_{\Delta_+} \odot b^{(j)}) + \sum_{g\in\Delta_+} (a^{(i)}_{\Delta_+} \odot b^{(i)}_g)^\dagger \cdot (\beta_g(a^{(j)}_{\Delta_u^c}) \odot b^{(j)}) + \\
&\quad + \sum_{g\in\Delta_-} (\beta_g(a^{(i)}) \odot b^{(i)}_g)^\dagger \cdot (\beta_g(a^{(j)}_{\Delta_-}) \odot b^{(j)}) + \sum_{g\in\Delta_+} (\beta_g(a^{(i)}_{\Delta_-}) \odot b^{(i)}_g)^\dagger \cdot (\beta_g(a^{(j)}_{\Delta_-}) \odot b^{(j)})
\end{aligned}$$

and taking into account that $\mathrm{spt}(\varphi), \mathrm{spt}(\omega) \in [\mathfrak{B}_\sigma \colon \sigma \in \Delta_+]$

$$\begin{aligned}
&(\omega \times \varphi)\left(\left[\left(a^{(i)}_{\Delta_+} + a^{(i)}_{\Delta_-}\right) \odot \left(b^{(i)}_{\Delta_+} + b^{(i)}_{\Delta_-}\right)\right]^* \left[\left(a^{(j)}_{\Delta_+} + a^{(j)}_{\Delta_-}\right) \odot \left(b^{(j)}_{\Delta_+} + b^{(j)}_{\Delta_-}\right)\right]\right) = \\
&= \psi_{\omega,\varphi}\left[\left(a^{(i)}_{\Delta_+} \odot b^{(i)}_{\Delta_+}\right)^\dagger \cdot \left(a^{(j)}_{\Delta_+} \odot b^{(j)}_{\Delta_+}\right)\right] + \psi_{\omega,\varphi}\left[\sum_{g\in\Delta_-} \left(\beta_g\left(a^{(i)}_{\Delta_+}\right) \odot b^{(i)}_g\right)^\dagger \cdot \left(\beta_g\left(a^{(j)}_{\Delta_+}\right) \odot b^{(j)}_{\Delta_-}\right)\right] + \\
&+ \psi_{\omega,\varphi}\left[\sum_{g\in\Delta_-} \left(\beta_g\left(a^{(i)}_{\Delta_-}\right) \odot b^{(i)}_g\right)^\dagger \cdot \left(\beta_g\left(a^{(j)}_{\Delta_-}\right) \odot b^{(j)}_{\Delta_-}\right)\right] + \psi_{\omega,\varphi}\left[\left(a^{(i)}_{\Delta_-} \odot b^{(i)}_{\Delta_+}\right)^\dagger \cdot \left(a^{(j)}_{\Delta_-} \odot b^{(j)}_{\Delta_+}\right)\right] .
\end{aligned} \tag{III.19}$$

Now, for a fixed representative $\widetilde{g} \in \Delta_- = \{(1,0), (0,1)\}$,

- $\beta_g\left(a^{(i)}_{\Delta_+}\right) = \beta_{\widetilde{g}}\left(a^{(i)}_{\Delta_+}\right)$ for $g \in \Delta_- = \{(1,0), (0,1)\}$. Therefore, the second addend in Equation III.19 becomes

$$\psi_{\omega,\varphi}\left[\sum_{g\in\Delta_-}\left(\beta_g(a^{(i)}_{\Delta_+})\odot b^{(i)}_g\right)^\dagger\cdot\left(\beta_g(a^{(j)}_{\Delta_+})\odot b^{(j)}_{\Delta_-}\right)\right]=$$
$$=\psi_{\omega,\varphi}\left[\sum_{g\in\Delta_-}\left(\beta_{\widetilde{g}}(a^{(i)}_{\Delta_+})\odot b^{(i)}_g\right)^\dagger\cdot\left(\beta_{\widetilde{g}}(a^{(j)}_{\Delta_+})\odot b^{(j)}_{\Delta_-}\right)\right]=$$
$$=\psi_{\omega,\varphi}\left[\left(\sum_{g\in\Delta_-}\beta_{\widetilde{g}}(a^{(i)}_{\Delta_+})\odot b^{(i)}_g\right)^\dagger\cdot\left(\beta_{\widetilde{g}}(a^{(j)}_{\Delta_+})\odot b^{(j)}_{\Delta_-}\right)\right]=$$
$$=\psi_{\omega,\varphi}\left[\left(\beta_{g'}\left(a^{(i)}_{\Delta_+}\right)\odot b^{(i)}_{\Delta_-}\right)^\dagger\cdot\left(\beta_{g'}\left(a^{(j)}_{\Delta_+}\right)\odot b^{(j)}_{\Delta_-}\right)\right].$$

- $\beta_g\left(a^{(i)}_{\Delta_-}\right)=f_{g-\widetilde{g}}\beta_{\widetilde{g}}\left(a^{(i)}_{\Delta_-}\right)$ for $g\in\Delta_-=\{(1,0),(0,1)\}$. Therefore, the third addend in Equation III.19 becomes

$$\sum_{g\in\Delta_-}\left(f_{g-\widetilde{g}}\beta_{\widetilde{g}}\left(a^{(i)}_{\Delta_-}\right)\odot b^{(i)}_g\right)^\dagger\cdot\left(f_{g-\widetilde{g}}\beta_{\widetilde{g}}\left(a^{(j)}_{\Delta_-}\right)\odot b^{(j)}_{\Delta_-}\right)=$$
$$=\sum_{g\in\Delta_-}(f_{g-\widetilde{g}})^2\left(\beta_{\widetilde{g}}\left(a^{(i)}_{\Delta_-}\right)\odot b^{(i)}_g\right)^\dagger\cdot\left(\beta_{\widetilde{g}}\left(a^{(j)}_{\Delta_-}\right)\odot b^{(j)}_{\Delta_-}\right)=$$
$$=\left(\beta_{\widetilde{g}}\left(a^{(i)}_{\Delta_-}\right)\odot b^{(i)}_{\Delta_-}\right)^\dagger\cdot\left(\beta_{\widetilde{g}}\left(a^{(j)}_{\Delta_-}\right)\odot b^{(j)}_{\Delta_-}\right)$$

Putting all together (once fixed $\widetilde{g}\in\Delta_-=\{(1,0),(0,1)\}$),

$$(\omega\times\varphi)(x^*x)=\psi_{\omega,\varphi}\left[\left(\sum_{i=1}^n a^{(i)}_{\Delta_+}\odot b^{(i)}_{\Delta_+}\right)^\dagger\cdot\left(\sum_{j=1}^n a^{(j)}_{\Delta_+}\odot b^{(j)}_{\Delta_+}\right)\right]+$$
$$+\psi_{\omega,\varphi}\left[\left(\sum_{i=1}^n \beta_{\widetilde{g}}\left(a^{(i)}_{\Delta_+}\right)\odot b^{(i)}_{\Delta_-}\right)^\dagger\cdot\left(\sum_{j=1}^n \beta_{\widetilde{g}}\left(a^{(j)}_{\Delta_+}\right)\odot b^{(j)}_{\Delta_-}\right)\right]+$$
$$+\psi_{\omega,\varphi}\left[\left(\sum_{i=1}^n \beta_{\widetilde{g}}(a^{(i)}_{\Delta_-})\odot b^{(i)}_{\Delta_-}\right)^\dagger\cdot\left(\sum_{j=1}^n \beta_{\widetilde{g}}(a^{(j)}_{\Delta_-})\odot b^{(j)}_{\Delta_-}\right)\right]+$$
$$+\psi_{\omega,\varphi}\left[\left(\sum_{i=1}^n a^{(i)}_{\Delta_-}\odot b^{(i)}_{\Delta_+}\right)^\dagger\cdot\left(\sum_{j=1}^n a^{(j)}_{\Delta_-}\odot b^{(j)}_{\Delta_+}\right)\right]$$

All the four terms are manifestly positive, whence $\omega\times\varphi$ is a state on $\mathfrak{B}\circledR\mathfrak{B}$. The inequality in the assertion is nothing but the Cauchy–Bunyakovsky–Schwarz inequality. Lastly, by Lemma II.3.1, $\pi_{\omega\times\varphi}$ uniquely extends to a representation of $\mathfrak{B}\circledR\mathfrak{B}$ acting by bounded operators on the Hilbert space $\mathcal{H}_{\omega\times\varphi}$. □

Remark III.8.3

Observe that the last equality can also be re-written as

$$(\omega\times\varphi)(x^*x)=\psi_{\omega,\varphi}\left(x_+^\dagger\cdot x_+\right)+\psi_{\omega\circ\widetilde{g},\varphi}\left(x_-^\dagger\cdot x_-\right)$$

or

$$\|\pi_{\omega\times\varphi}(x)\xi_{\omega\times\varphi}\|_{\mathcal{H}_{\omega\times\varphi}}=\sqrt{\|(\pi_\omega\otimes\pi_\varphi)(x_+)(\xi_\omega\otimes\xi_\varphi)\|^2_{\mathcal{H}_\omega\otimes\mathcal{H}_\varphi}+\|(\pi_{\omega\circ\widetilde{g}}\otimes\pi_\varphi)(x_-)(\xi_{\omega\circ\widetilde{g}}\otimes\xi_\varphi)\|^2_{\mathcal{H}_{\omega\circ\widetilde{g}}\otimes\mathcal{H}_\varphi}}$$

where $\omega \circ \widetilde{g} \in \mathcal{S}(\mathfrak{B})$ and $x_\pm := \sum_{i=1}^{n} a^{(i)} \odot b^{(i)}_{\Delta_\pm}$.

In view of Lemma III.8.2, we can build an "intermediate" C^*-norm on $\mathfrak{B} \circledast \mathfrak{B}$ based on products of Δ_+-invariant states, which is compatible with the direct product action of $K_4 \times K_4$.

Theorem III.8.4
Let $(\mathfrak{B}, K_4, \beta)$ be a C^*-system. Then, $\|\cdot\|_{\Delta_+} := \sup_{\omega,\varphi \in \mathcal{S}_{\Delta_+}(\mathfrak{B})} \|\pi_{\omega\times\varphi}(\cdot)\|$ is a $(\beta\times\beta)$-compatible C^*-norm on $\mathfrak{B} \circledast \mathfrak{B}$. In particular, $\|x\|_{\min} \leq \|x\|_{\Delta_+} \leq \|x\|_{\max}$ $(x \in \mathfrak{B} \circledast \mathfrak{B})$, where the equalities are simultaneously satisfied for every $x \in \mathfrak{B} \circledast \mathfrak{B}$ if and only if $\mathfrak{B}^{K_4}$ is nuclear.

Proof.
Evidently $\mathcal{S}_{K_4}(\mathfrak{B}) \subset \mathcal{S}_{\Delta_+}(\mathfrak{B})$[4], hence $\mathcal{S}_{\Delta_+}(\mathfrak{B}) \times \mathcal{S}_{\Delta_+}(\mathfrak{B})$ separates the points of $\mathfrak{B} \circledast \mathfrak{B}$ (cfr. Proposition II.9.3) and $\|\cdot\|_{\Delta_+}$ defines a C^*-norm on $\mathfrak{B} \circledast \mathfrak{B}$. Now, if $\omega \in \mathcal{S}_{\Delta_+}(\mathfrak{B})$, then $\omega \circ \beta_g \in \mathcal{S}_{\Delta_+}(\mathfrak{B})$ for every $g \in K_4$. It follows that $\mathcal{S}_{\Delta_+}(\mathfrak{B}) \times \mathcal{S}_{\Delta_+}(\mathfrak{B})$ is left globally stable by the transposed action $(\beta\times\beta)^{\mathrm{t}}$ of $K_4 \times K_4$. By Theorem II.10.5, $\|\cdot\|_{\Delta_+}$ is $(\beta\times\beta)$-compatible. Lastly, by Theorem II.12.3 and the very definition of the maximal C^*-norm,

$$\|x\|_{\min} \leq \|x\|_{\Delta_+} \leq \|x\|_{\max}, \quad x \in \mathfrak{B} \circledast \mathfrak{B}\,,$$

where the equalities are simultaneously satisfied for every $x \in \mathfrak{B} \circledast \mathfrak{B}$ if and only if $\mathfrak{B}^{K_4}$ is nuclear, thanks to Theorem II.17.4. □

Remark III.8.5
We remark that the C^*-algebra $\mathfrak{B}$ in the previous theorem has well a $\mathbb{Z}_2$-graded structure induced by the restriction of the action β to any of the three order-2 subgroups of K_4, $N_1 := \langle(1,1)\rangle = \Delta_+$, $N_2 := \langle(1,0)\rangle$ and $N_3 := \langle(0,1)\rangle$. Nevertheless, the involutive algebra $(\mathfrak{B}, K_4, \beta) \circledast (\mathfrak{B}, K_4, \beta)$ is in general not isomorphic to any of the algebras $(\mathfrak{B}, N_{i_1}, \beta|_{N_{i_1}}) \circledast_F (\mathfrak{B}, N_{i_2}, \beta|_{N_{i_2}})$, $i_1, i_2 \in \{1,2,3\}$. Indeed, it is easy to verify that, if $a, b \in \mathfrak{B}$ are homogeneous,

$$(a \circledast b)^{*,\mathrm{K}} = (-1)^{\partial_{K_4}(a)\cdot\partial_{K_4}(b)} = -a^* \circledast b^*$$

if and only if $\partial_{K_4}(a) \cdot \partial_{K_4}(b) = 1$, which happens in exactly *six* instances. On the other hand,

$$(a \circledast_F b)^{*,\mathrm{F}} = (-1)^{\partial_{N_{i_1}}(a)\cdot\partial_{N_{i_2}}(b)} = -a^* \circledast_F b^*$$

in exactly *four* cases, for any $i_1, i_2 \in \{1,2,3\}$. It follows that in general the identity map

$$I_{\mathfrak{B}\odot\mathfrak{B}} \colon (\mathfrak{B}, K_4, \beta) \circledast (\mathfrak{B}, K_4, \beta) \to (\mathfrak{B}, N_{i_1}, \beta|_{N_{i_1}}) \circledast_F (\mathfrak{B}, N_{i_2}, \beta|_{N_{i_2}})$$

is not selfadjoint, let alone a *-isomorphism of involutive algebras.

Without assuming the nuclearity of $\mathfrak{B}$, it seems unclear whether $\|\cdot\|_{\min}$ coincides with $\|\cdot\|_{\Delta_+}$ or not, since in general $\mathcal{S}_{\Delta_+}(\mathfrak{B})$ can properly contain $\mathcal{S}_{K_4}(\mathfrak{B})$. Still, when $\mathfrak{B}$ is nuclear, in view of Lemma III.8.2 and Theorem III.8.4, we can construct the *infinite product state* of a Klein twisted chain $(\mathfrak{A}, K_4, \alpha)$ generated by a sequence $(\psi_i)_{i\in\mathbb{N}} \subset \mathcal{S}_{\Delta_+}(\mathfrak{B})$. Indeed, let

$$\begin{cases} \omega_1 := \psi_1 \in \mathcal{S}_{\Delta_+}(\mathfrak{B}) \\ \omega_{n+1} := \omega_n \times \psi_{n+1} \in \mathcal{S}_{\Delta_+}(\mathfrak{A}_{n+1}), \quad n \in \mathbb{N} \end{cases}$$

[4] Even more: if $\mathfrak{B}_{(1,1)} \neq \{0\}$, then $\mathcal{S}_{K_4}(\mathfrak{B}) \subsetneq \mathcal{S}_{\Delta_+}(\mathfrak{B})$. Indeed, let $x \in \mathfrak{B}_{(1,1)}$ be a non-zero element. Then, either $x + x^* \in \mathfrak{B}_{(1,1)}$ or $i(x - x^*) \in \mathfrak{B}_{(1,1)}$ is non-zero. Since for every normal element $b \in \mathfrak{B}$, there must exist a state $\omega \in \mathcal{S}(\mathfrak{B})$ s.t. $|\omega(b)| = \|b\|$, we conclude that there exists a state $\varphi \in \mathcal{S}(\mathfrak{B})$ s.t. $\varphi|_{\mathfrak{B}_{(1,1)}} \neq 0$. In particular, $\varphi \circ (E_{(0,0)} + E_{(1,1)}) \in \mathcal{S}_{\Delta_+}(\mathfrak{B}) \setminus \mathcal{S}_{K_4}(\mathfrak{B})$.

where $\mathfrak{A}_{n+1} = \mathfrak{A}_n \circledast_{\min} \mathfrak{B} = \mathfrak{A}_n \circledast_{\Delta_+} \mathfrak{B}$ (the last equality being given by the nuclearity of $\mathfrak{B}$). Notice that the sequence $(\omega_n)_{n\in\mathbb{N}}$ evidently satisfies the relations

$$\omega_{n+1} \circ \iota_n = \omega_n , \quad n \in \mathbb{N}$$

so that we can define a (algebraically) positive, unital, linear functional $\omega_\infty \colon \mathfrak{A}_\infty \to \mathbb{C}$ by

$$\omega_\infty(a) := \omega_n(a_n)$$

for every $a \in \mathfrak{A}_\infty$, $n \in \mathbb{N}$ and $a_n \in \mathfrak{A}_n$ that satisfy $\phi_n(a_n) = a$. Moreover, $|\omega_\infty(a)| \leq \|a_n\|_{\mathfrak{A}_n} = \|a\|_{\mathfrak{A}}$ hence ω_∞ extends to a well-defined state ω on $\mathfrak{A}$, the unique one satisfying

$$\omega\left(j_1(b_1)\dots j_n(b_n)\right) = \prod_{i=1}^{n} \psi_i(b_i)$$

for every $b_i \in \mathfrak{B}_i = \mathfrak{B}$, $i = 1, \dots, n$ $(n \in \mathbb{N})$. Plus, ω is invariant under the restriction to Δ_+ of the K_4-action on $\mathfrak{A}$. By denoting ω with $\prod_{n\in\mathbb{N}} \psi_n \in \mathcal{S}_{\Delta_+}(\mathfrak{A})$, we collect the result of the above construction in the following

Corollary III.8.6
Let $(\mathfrak{B}, K_4, \beta)$ be a C^*-system, with $\mathfrak{B}$ nuclear. If $(\psi_n)_{n\in\mathbb{N}} \subset \mathcal{S}_{\Delta_+}(\mathfrak{B})$, then the infinite product functional $\omega := \prod_{n\in\mathbb{N}} \psi_n$ is a well-defined, Δ_+-invariant state of the (minimal) Klein twisted chain $(\mathfrak{A}, K_4, \alpha)$ of $\mathfrak{B}$.

Of crucial importance for the De Finetti theorem on a Klein twisted C^*-chain are, again, the infinite product states, now of form $\prod_{n\in\mathbb{N}} \psi$ for some fixed $\psi \in \mathcal{S}_{\Delta_+}(\mathfrak{B})$. As expected, they will be exactly the ergodic symmetric states of the case in question. We follow the path traced by Størmer, as already done by Fidaleo in the Fermi case.

Lemma III.8.7
Let $\omega \in \mathcal{E}_{\mathfrak{S}}(\mathfrak{A})$. Then, $\underset{n\to+\infty}{w\text{-lim}}(\pi_\omega(\rho_n(a))\xi_\omega) = \omega(a)\xi_\omega \quad (a \in \mathfrak{A})$.

Proof.
If $a \in \mathfrak{A}_\infty$ and $\rho \in \mathfrak{S}$, then there exists $N_{a,\rho}$ s.t. for every $n \geq N_{a,\rho}$ $\rho(\rho_n(a)) = \rho_n(a)$. Let $\mathcal{D}$ be the derived set of $\{\pi_\omega(\rho_n(a))\xi_\omega\}_n \subset B^{\mathcal{H}_\omega}_{\|a\|}$ in the weak topology on $\mathcal{H}_\omega$. By weak (equivalently, weak sequential) compactness of $B^{\mathcal{H}_\omega}_{\|a\|}$, $\mathcal{D} \neq \varnothing$. Let $\xi \in \mathcal{D}$, i.e. there exists $\{\rho_{n_k}\}_k \subset \{\rho_n\}_n$ s.t. $\xi = \underset{k\to+\infty}{w\text{-lim}}(\pi_\omega(\rho_{n_k}(a))\xi_\omega)$. Then, for every $\rho \in \mathfrak{S}$,

$$U_\omega(\rho)\xi = \underset{k\to+\infty}{w\text{-lim}}(U_\omega(\rho)\pi_\omega(\rho_{n_k}(a))\xi_\omega) = \underset{k\to+\infty}{w\text{-lim}}(\pi_\omega(\rho(\rho_{n_k}(a)))U_\omega(\rho)\xi_\omega) = \underset{k\to+\infty}{w\text{-lim}}(\pi_\omega(\rho_{n_k}(a))\xi_\omega) = \xi$$

that is $\xi \in \mathcal{H}^{\mathfrak{S}}_\omega$. Since $\omega \in \mathcal{E}_{\mathfrak{S}}(\mathfrak{A})$ is $\mathfrak{S}$-abelian, $\mathcal{H}^{\mathfrak{S}}_\omega = \mathbb{C}\xi_\omega$ so that $\xi = c\xi_\omega$ for some $c \in \mathbb{C}$. Precisely,

$$c = \langle \xi, \xi_\omega \rangle = \lim_{k\to+\infty} \langle \pi_\omega(\rho_{n_k}(a))\xi_\omega, \xi_\omega \rangle = \langle \pi_\omega(a)\xi_\omega, \xi_\omega \rangle = \omega(a)\,.$$

It follows that $\mathcal{D} = \{\omega(a)\xi_\omega\}$. Hence, $\{\pi_\omega(\rho_n(a))\xi_\omega\}_n$ is weakly convergent and

$$\underset{n\to+\infty}{w\text{-lim}}(\pi_\omega(\rho_n(a))\xi_\omega) = \omega(a)\xi_\omega\,.$$

Lastly, for general $a \in \mathfrak{A}$ and any $\varepsilon > 0$, there exists $a_\varepsilon \in \mathfrak{A}_\infty$ s.t.

$$|\langle \pi_\omega(\rho_n(a - a_\varepsilon))\xi_\omega, \eta \rangle| < \varepsilon$$

for every $n \geq 1$, $\eta \in B_1^{\mathcal{H}_\omega}$. Since $\{\langle \pi_\omega(\rho_n(a_\varepsilon))\xi_\omega, \eta\rangle\}_n$ converges to $\omega(a_\varepsilon)\langle \xi_\omega, \eta\rangle$, $\langle \pi_\omega(\rho_n(a))\xi_\omega, \eta\rangle$ is a Cauchy sequence, hence convergent too. Precisely, by continuity of $\omega \in \mathcal{E}_{\mathfrak{S}}(\mathfrak{A})$,

$$\lim_{n\to+\infty} \langle \pi_\omega(\rho_n(a))\xi_\omega, \eta\rangle = \lim_{\varepsilon\downarrow 0^+} \left(\lim_{n\to+\infty} \langle \pi_\omega(\rho_n(a_\varepsilon))\xi_\omega, \eta\rangle \right) = \lim_{\varepsilon\downarrow 0^+} \omega(a_\varepsilon)\langle \xi_\omega, \eta\rangle = \omega(a)\langle \xi_\omega, \eta\rangle$$

whence $\underset{n\to+\infty}{w\text{-}\lim}(\pi_\omega(\rho_n(a))\xi_\omega) = \omega(a)\xi_\omega$. □

Theorem III.8.8 (Characterization of $\mathcal{E}_{\mathfrak{S}}(\mathfrak{A})$, $\mathfrak{A}$ Klein C^*-chain)
Let $(\mathfrak{B}, K_4, \beta)$ be a C^*-system, with $\mathfrak{B}$ nuclear, and $\mathfrak{A}$ its associated Klein chain. If $\omega \in \mathcal{S}_{\mathfrak{S}}(\mathfrak{A})$, the following are equivalent:

(i) $\omega \in \mathcal{E}_{\mathfrak{S}}(\mathfrak{A})$

(ii) ω is strongly clustering

(iii) ω is weakly clustering in average

(iv) $\omega = \prod_{n\in\mathbb{N}} \psi$ for some $\psi \in \mathcal{S}_{\Delta_+}(\mathfrak{B}) \cong \mathcal{S}(\mathfrak{B}_{(0,0)} \oplus \mathfrak{B}_{(1,1)})$

In particular, $\mathcal{E}_{\mathfrak{S}}(\mathfrak{A}) = \left\{ \prod_{n\in\mathbb{N}} \psi \right\}_{\psi\in\mathcal{S}_{\Delta_+}(\mathfrak{B})}$ is weakly-* closed, thus making $\mathcal{S}_{\mathfrak{S}}(\mathfrak{A})$ a Bauer simplex in $\mathcal{S}(\mathfrak{A})$. Precisely, $(\mathcal{S}_{\Delta_+}(\mathfrak{B}), \tau_{w^*})$ and $(\mathcal{E}_{\mathfrak{S}}(\mathfrak{A}), \tau_{w^*})$ are homeomorphic via the mapping

$$\begin{aligned} \iota\colon \mathcal{S}_{\Delta_+}(\mathfrak{B}) &\to \mathcal{E}_{\mathfrak{S}}(\mathfrak{A}) \\ \psi &\to \prod_{n\in\mathbb{N}} \psi \end{aligned}$$

Lastly, there exists a $\mathfrak{S}$-invariant, densely ranged p.u. map of C^*-algebras

$$T\colon \mathfrak{A} \to \mathcal{C}(\mathcal{S}_{\Delta_+}(\mathfrak{B}))$$

s.t. its transpose $T^{\mathrm{t}}\colon \mathcal{M}_1(\mathcal{S}_{\Delta_+}(\mathfrak{B})) \to \mathcal{S}_{\mathfrak{S}}(\mathfrak{A})$ is an affine homeomorphism.

Proof.
(i) ⇒ (ii) Suppose $\omega \in \mathcal{E}_{\mathfrak{S}}(\mathfrak{A})$, and take $a, b \in \mathfrak{A}$. Then, by Lemma III.8.7,

$$\lim_{n\to+\infty} \omega(a\rho_n(b)) = \lim_{n\to+\infty} \langle \pi_\omega(\rho_n(b))\xi_\omega, \pi_\omega(a^*)\xi_\omega\rangle = \omega(a)\omega(b),$$

that is ω is strongly clustering.
(ii) ⇒ (i) Choose a vector $\xi \in \mathcal{H}_\omega^{\mathfrak{S}}$ s.t. $\xi \perp \xi_\omega$. We shall show that $\xi = 0$. Fix $\varepsilon > 0$. By cyclicity of ξ_ω, there exists $b \in \mathfrak{A}$ such that $\|\xi - \pi_\omega(b)\xi_\omega\| \leq \varepsilon/2$, and thus

$$|\omega(b)| = |\langle \pi_\omega(b)\xi_\omega, \xi_\omega\rangle| = |\langle(\pi_\omega(b)\xi_\omega - \xi), \xi_\omega\rangle| \leq \varepsilon/2\,.$$

Let now $a \in \mathfrak{A}$ such that $\|\pi_\omega(a)\| \leq 1$. Recalling that ξ and ξ_ω are both $U_\omega(\mathfrak{S})$-invariant, we get

$$\begin{aligned} |\langle \xi, \pi_\omega(a)\xi_\omega\rangle| &= |\langle U_\omega(\rho_n)\pi_\omega(a^*)U_\omega(\rho_n^{-1})\xi, \xi_\omega\rangle| \leq \\ &\leq |\langle U_\omega(\rho_n)\pi_\omega(a^*)U_\omega(\rho_n^{-1})\pi_\omega(b)\xi_\omega, \xi_\omega\rangle| + \varepsilon/2 = |\omega(\rho_n(a^*)b)| + \varepsilon/2. \quad \text{(III.20)} \end{aligned}$$

By assumption ω is strongly clustering, thus as n tends to $+\infty$ on both sides of Equation III.20, we get

$$|\langle \xi, \pi_\omega(a)\xi_\omega\rangle| \leq |\omega(a^*)||\omega(b)| + \varepsilon/2 \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

By arbitrariness of $\varepsilon > 0$ and cyclicity of ξ_ω for $\pi_\omega(\mathfrak{A})$, we get $\xi = 0$. It follows that $\mathcal{H}_\omega^{\mathfrak{G}} = \mathbb{C}\xi_\omega$, whence $\omega \in \mathcal{E}_{\mathfrak{G}}(\mathfrak{A})$.

(iv) $\Rightarrow$ (ii) It suffices to show the implication for $a := \textcircled{\scriptsize K}_{j\in\mathbf{t}} a_j, b := \textcircled{\scriptsize K}_{k\in\mathbf{u}} b_k$, $t, u \geq 1$ (here, we have identified a, b with their images in $\mathfrak{A}$ under the canonical embeddings Φ_t, Φ_u, respectively). For every $n \geq 1$,

$$\omega\left(a\rho_n(b)\right) = \prod_{j=1}^{2^{n-1}} \psi(a_j b_{j+2^{n-1}}) \prod_{j=2^{n-1}+1}^{2^n} \psi(a_j b_{j-2^{n-1}}) \prod_{j\geq 2^n+1} \psi(a_j b_j)$$

hence for every $n > \lceil \log_2(\max\{t, u\}) \rceil$

$$\omega\left(a\rho_n(b)\right) = \prod_{j=1}^{2^{n-1}} \psi(a_j) \prod_{j=2^{n-1}+1}^{2^n} \psi(b_{j-2^{n-1}}) = \prod_{j=1}^{2^{n-1}} \psi(a_j) \prod_{j=1}^{2^{n-1}} \psi(b_j) = \omega(a)\omega(b)\,,$$

that is the sequence $\{\omega(a\rho_n(b))\}_n \subset \mathbb{C}$ is definitely constant. A fortiori, $\lim_{n\to+\infty} \omega\left(a\rho_n(b)\right) = \omega(a)\omega(b)$ and ω is strongly clustering.

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i) comes from Theorem III.2.1.

(ii) $\Rightarrow$ (iv) For each $n \geq 1$, we consider the embedding

$$\mathfrak{B} \ni a \mapsto j_n(a) := \underbrace{\mathbb{1} \,\textcircled{\scriptsize K} \cdots \textcircled{\scriptsize K}\, \mathbb{1}}_{n \text{ times}} \,\textcircled{\scriptsize K}\, a \,\textcircled{\scriptsize K}\, \mathbb{1} \,\textcircled{\scriptsize K} \cdots \in \mathfrak{A}\,.$$

To prove (iv) it suffices to show that, for each $n \geq 1$ and $a_1, \dots, a_n \in \mathfrak{B}$,

$$\omega\big(j_1(a_1)\dots j_n(a_n)\big) = \prod_{i=1}^{n} \psi(a_i) \tag{III.21}$$

for some state $\psi \in \mathcal{S}_{\Delta_+}(\mathfrak{B})$. Firstly, thanks to point (i) in Corollary III.7.2, $\psi_k := \omega\circ j_k \in \mathcal{S}_{\Delta_+}(\mathfrak{B})$ ($k \geq 1$). Moreover, since $\omega \in \mathcal{S}_{\mathfrak{G}}(\mathfrak{A})$, $\psi_k = \psi_l =: \psi \in \mathcal{S}_{\Delta_+}(\mathfrak{B})$ for every pair $k, l \geq 1$. We now shall prove (III.21) by an induction procedure on $n \geq 1$. For $n = 1$, $\omega \circ j_1 = \psi$ by the very definition. We then assume (III.21) true for some fixed $n \geq 1$ and prove it for $n + 1$. To accomplish that, notice that for any $m > n + 1$, if $\rho \in \mathfrak{G}$ is such that $\rho|_{\mathbf{n}} = \mathrm{id}_{\mathbf{n}}$ and $\rho(n + 1) = \rho_m(n + 1)$ (where ρ_m is defined in Equation III.14), then

$$\begin{aligned}\omega\big(j_1(a_1)\dots j_n(a_n)j_{n+1}(a_{n+1})\big) &= (\omega\circ\rho)\big(j_1(a_1)\dots j_n(a_n)j_{n+1}(a_{n+1})\big) = \\ &= \omega\big(j_1(a_1)\dots j_n(a_n)\rho_m(j_{n+1}(a_{n+1}))\big)\,.\end{aligned} \tag{III.22}$$

If $\varepsilon > 0$, by strong clustering property of ω there exists $m_\varepsilon > n + 1$ s.t.

$$\begin{aligned}&\big|\omega\big(j_1(a_1)\dots j_n(a_n)\rho_{m_\varepsilon}(j_{n+1}(a_{n+1}))\big) - \omega\big(j_1(a_1)\dots j_n(a_n)\big)\omega(j_{n+1}(a_{n+1}))\big| = \\ &\qquad = \big|\omega\big(j_1(a_1)\dots j_n(a_n)\rho_{m_\varepsilon}(j_{n+1}(a_{n+1}))\big) - \omega\big(j_1(a_1)\dots j_n(a_n)\big)\psi(a_{n+1})\big| \leq \varepsilon\end{aligned}$$

that is, by inductive hypothesis,

$$\left|\omega\big(j_1(a_1)\dots j_n(a_n)\rho_{m_\varepsilon}(j_{n+1}(a_{n+1}))\big) - \prod_{i=1}^{n+1}\psi(a_i)\right| \leq \varepsilon\,.$$

Recalling Equation III.22, $\big|\omega\big(j_1(a_1)\dots j_n(a_n)j_{n+1}(a_{n+1})\big) - \omega\big(j_1(a_1)\dots j_n(a_n)\big)\psi(a_{n+1})\big| \leq \varepsilon$. The assertion now follows, as $\varepsilon > 0$ is arbitrary. The rest of the theorem can be shown by following *verbatim* the proofs of Theorem 2.8 in [76] (p. 58) and Theorem 3.9 in [77] (p. 16). □

Remark III.8.9
From Theorem III.8.8 and Theorem III.2.9, $\mathcal{S}_{\mathfrak{G}}(\mathfrak{A})$, $\mathcal{M}_1(\mathcal{S}_{\Delta_+}(\mathfrak{B}))$ and $\mathcal{F}_{\rtimes} := \{\varphi \in \mathcal{S}(\mathfrak{A} \rtimes_{\alpha,\mathrm{r}} \mathfrak{G})\colon \varphi(u_\rho a u_\sigma) = \varphi(a), \quad \rho, \sigma \in \mathfrak{G}, a \in \mathfrak{A}\}$ are all affinely homeomorphic via

$$\begin{aligned} \mathcal{M}_1(\mathcal{S}_{\Delta_+}(\mathfrak{B})) \to \qquad & \mathcal{S}_{\mathfrak{G}}(\mathfrak{A}) \qquad \to \mathcal{F}_{\rtimes} \\ \mu \mapsto \int\limits_{\mathcal{S}_{\Delta_+}(\mathfrak{B})} \left(\prod_{n\in\mathbb{N}} \psi\right)(\cdot)\mathrm{d}\mu(\psi) & \mapsto \left[u_\rho a u_\sigma \mapsto \int\limits_{\mathcal{S}_{\Delta_+}(\mathfrak{B})} \left(\prod_{n\in\mathbb{N}} \psi\right)(a)\mathrm{d}\mu(\psi)\right] \end{aligned}$$

Recall that $\mathfrak{G}$ is amenable, hence the full and the reduced crossed products of $(\mathfrak{A}, \mathfrak{G}, \alpha)$ are isomorphic C^*-algebras: $\mathfrak{A} \rtimes_{\alpha,\mathrm{f}} \mathfrak{G} \cong \mathfrak{A} \rtimes_{\alpha,\mathrm{r}} \mathfrak{G}$. Similarly,

$$\mathcal{S}_{\Delta_+}(\mathfrak{B}) \cong \{\varphi \in \mathcal{S}(\mathfrak{B} \rtimes_{\alpha,\mathrm{r}} \Delta_+)\colon \varphi(u_{(1,1)}a) = \varphi(a u_{(1,1)}) = \varphi(a), \quad a \in \mathfrak{A}\}$$

We are now in position to establish the De Finetti theorem for Klein twisted C^*-chains, thus obtaining that any symmetric state is the mixture of product states, being each of them the product of a single Δ_+-invariant state.

Theorem III.8.10 (De Finetti theorem for Klein twisted C^*-chains)
Let $(\mathfrak{A}, \mathfrak{G})$ be the C^*-system associated to a unital, K_4-graded, nuclear C^*-algebra $\mathfrak{B}$, as in Theorem III.8.1. Then, for each $\varphi \in \mathcal{S}_{\mathfrak{G}}(\mathfrak{A})$, there exists a unique $\prec$-maximal $\mu_\varphi \in \mathcal{M}_1(\mathcal{S}_{\mathfrak{G}}(\mathfrak{A}))$ s.t.

$$\varphi(a) = \int\limits_{\mathcal{S}_{\mathfrak{G}}(\mathfrak{A})} \omega(a)\,\mathrm{d}\mu_\varphi(\omega), \qquad a \in \mathfrak{A}\,. \tag{III.23}$$

In particular, μ_φ is pseudo-supported by $\mathcal{E}_{\mathfrak{G}}(\mathfrak{A}) = \left\{\prod_{n\in\mathbb{N}} \psi\right\}_{\psi\in\mathcal{S}_{\Delta_+}(\mathfrak{B})}$ i.e. $\mu_\varphi(B) = 1$ for every $B \in \mathcal{B}_0(\mathcal{S}_{\mathfrak{G}}(\mathfrak{A}))$ containing $\mathcal{E}_{\mathfrak{G}}(\mathfrak{A})$. The relative weak-* topology on the unit ball $B_{\mathfrak{A}^*}$ of $\mathfrak{A}^*$ is metrizable if and only if $\mathfrak{B}$ is separable, in which case μ_φ is *supported* by $\mathcal{E}_{\mathfrak{G}}(\mathfrak{A})$ and Equation III.23 becomes

$$\varphi(a) = \int\limits_{\mathcal{E}_{\mathfrak{G}}(\mathfrak{A})} \omega(a)\,\mathrm{d}\mu_\varphi(\omega), \qquad a \in \mathfrak{A}\,. \tag{III.24}$$

We conclude this section by reminding the reader that all the ergodic analysis done here essentially relies on Theorem III.7.1, which in turn is built upon the combinatorial estimate of Lemma III.6.3. Since the estimate holds for the finitary *alternating* subgroup $\mathcal{A}$ too, we see that every $\mathcal{A}$-invariant state $\omega \in \mathcal{S}_{\mathcal{A}}(\mathfrak{A})$ on a Klein (or also Fermi) C^*-chain $\mathfrak{A}$ must still be Δ_+-invariant and $\mathcal{A}$-abelian. In particular, $\mathcal{S}_{\mathcal{A}}(\mathfrak{A})$ is a Choquet simplex as well, containing $\mathcal{S}_{\mathfrak{G}}(\mathfrak{A})$. Moreover, $\mathcal{A}$ is normal and has index two in $\mathfrak{G}$, so that $\mathfrak{G}/\mathcal{A} \cong \mathbb{Z}_2$. Then, if $\varphi \in \mathcal{S}_{\mathcal{A}}(\mathfrak{A})$, $\varphi \circ \rho = \varphi \circ \alpha_{(1\,2)}$ for every odd $\rho \in \mathfrak{G}$, where $(1\,2)$ is the first adjacent transposition of $\mathbb{N}$. We can then exploit Theorem 4.3.37 in [86] (p. 424) to give an ergodic decomposition of $\varphi \in \mathcal{E}_{\mathfrak{G}}(\mathfrak{A})$.

Theorem III.8.11 (Barycentric decomposition of $\mathcal{E}_{\mathfrak{G}}(\mathfrak{A})$ w.r.t. $\mathcal{A}$)
Let $\psi \in \mathcal{S}_{\Delta_+}(\mathfrak{B})$ and $\varphi := \prod_{n\in\mathbb{N}} \psi \in \mathcal{E}_{\mathfrak{G}}(\mathfrak{A})$. Then, there exists a unique $\prec$-maximal $\mu_\varphi \in \mathcal{M}_1(\mathcal{S}_{\mathcal{A}}(\mathfrak{A}))$ s.t.

$$\varphi(a) = \int\limits_{\mathcal{S}_{\mathcal{A}}(\mathfrak{A})} \omega(a)\mathrm{d}\mu_\varphi(\omega), \quad a \in \mathfrak{A}\,.$$

Furthermore,

- μ_φ is supported by a closed subset of $\mathcal{E}_{\mathcal{A}}(\mathfrak{A})$
- μ_φ coincides with the orthogonal measure associated to the abelian von Neumann algebra $\pi_\varphi(\mathfrak{A})' \cap U_\varphi(\mathcal{A})'$, i.e.
$$\pi_\varphi(\mathfrak{A})' \cap U_\varphi(\mathcal{A})' = \mathrm{im}(\kappa_{\mu_\varphi})$$
where $\kappa_{\mu_\varphi} \colon L^\infty_{\mu_\varphi}(\mathcal{S}_{\mathcal{A}}(\mathfrak{A})) \hookrightarrow \pi_\varphi(\mathfrak{A})'$ is the *Tomita's isomorphism*, the unique linear map satisfying
$$\left\langle \kappa_{\mu_\varphi}(f)\pi_\varphi(a)\xi_\varphi, \xi_\varphi \right\rangle_{\mathcal{H}_\varphi} = \int\limits_{\mathcal{S}_{\mathcal{A}}(\mathfrak{A})} f(\omega)\omega(a)\mathrm{d}\mu_\varphi(\omega), \quad f \in L^\infty_{\mu_\varphi}(\mathcal{S}_{\mathcal{A}}(\mathfrak{A})),\ a \in \mathfrak{A}$$
- there exists $\widetilde{\varphi} \in \mathcal{E}_{\mathcal{A}}(\mathfrak{A})$ s.t. $\mu_\varphi(f) = \dfrac{f(\widetilde{\varphi}) + f(\widetilde{\varphi} \circ \alpha_{(1\,2)})}{2}$, $f \in \mathcal{C}(\mathcal{S}_{\mathcal{A}}(\mathfrak{A}))$, and in particular $\varphi(a) = \dfrac{\widetilde{\varphi}(a) + \widetilde{\varphi} \circ \alpha_{(1\,2)}(a)}{2}$, $a \in \mathfrak{A}$.

III.9. Some models of Klein chains

III.9.1 Continuous functions on the circle

Given a unital C^*-algebra $\mathfrak{B}$, a faithful action $K_4 \overset{\beta}{\curvearrowright} \mathfrak{B}$ is equivalent to a pair of commuting involutive automorphisms (briefly, involutions) $\vartheta, \tau \in \mathsf{Aut}(\mathfrak{B})$ s.t. $\vartheta, \tau \neq I_{\mathfrak{B}}$, via the assignment $\beta_{(1,0)} = \vartheta$, $\beta_{(0,1)} = \tau$. If $\mathfrak{B} = \mathcal{C}(X)$ with X compact Hausdorff space, each involution of $\mathfrak{B}$ is uniquely determined by an involutive, continuous self-map on X. For $X = \mathbb{T}$, involutions of a circle where thoroughly examined by Pfeffer in two instructive brief papers, [64] and [65]. We collect here the fundamental results in [65], and enrich them with some (easy) facts from algebraic topology. For $\vartheta \in \mathsf{Homeo}(\mathbb{T})$, let $\mathrm{w}(\vartheta)$ be its *winding number* (informally, the number of times its oriented image "wraps around" the center of $\mathbb{T}$). Also, let $R_\pi \in \mathsf{Homeo}(\mathbb{T})$ be the rotation by an angle of π radiants and $r_{\mathrm{Re}}, r_{\mathrm{Im}} \in \mathsf{Homeo}(\mathbb{T})$ the reflections through the real and the imaginary axes of the Argand-Gauss plane respectively.

Proposition III.9.1 (Pfeffer, 1974)
Let $\vartheta \in \mathsf{Homeo}(\mathbb{T})$ be s.t. $\vartheta \neq I_{\mathbb{T}}$ and $\vartheta^2 = I_{\mathbb{T}}$. Then,

(1) ϑ is not minimal

(2) $\mathrm{w}(\vartheta) \in \{\pm 1\}$: $\mathrm{w}(\vartheta) = 1$ ($\mathrm{w}(\vartheta) = -1$) iff ϑ is orientation-preserving (reversing), in which case it is homotopic to $I_{\mathbb{T}}$ (r_{Re})

(3) $|\mathrm{Fix}(\vartheta)| \in \{0, 2\}$: $|\mathrm{Fix}(\vartheta)| = 0$ iff ϑ is free; $|\mathrm{Fix}(\vartheta)| = 2$ iff $\vartheta \circ \tau = \tau \circ \vartheta$ for some free involution $\tau \neq \vartheta$

(4) for every free involution τ, there exists $z \in \mathbb{T}$ s.t. $\vartheta(z) = \tau(z)$ (in particular, there exists $z \in \mathbb{T}$ s.t. $\vartheta(z) = -z$)

(5) $B_0(1) = \bigcup\limits_{z \in \mathbb{T}} (z, \vartheta(z))$

Moreover, if τ_1, τ_2 are free distinct involutions, $\tau_1 \circ \tau_2 \neq \tau_2 \circ \tau_1$. Lastly,
$$\{\vartheta \in \mathsf{Homeo}(\mathbb{T}) \colon \vartheta^2 = I_{\mathbb{T}}, \mathrm{w}(\vartheta) = 1\} \subset \mathrm{Cl}(R_\pi)$$
$$\{\vartheta \in \mathsf{Homeo}(\mathbb{T}) \colon \vartheta^2 = I_{\mathbb{T}}, \mathrm{w}(\vartheta) = -1\} \subset \mathrm{Cl}(r_{\mathrm{Re}})$$
where Cl denotes the conjugacy class in the group $\mathsf{Homeo}(\mathbb{T})$.

Proof.
As concerns point (1), for each $z \in \mathbb{T}$, $T_z := \{z, \vartheta(z)\}$ is a non-trivial, closed, ϑ-invariant set.
Point (2) is a basic result in algebraic topology.
Point (3) is Corollary 2 and Proposition 1 (p. 614).
Point (4) is Proposition 2 and Corollary 3 (p. 615).
Point (5) is Corollary 4 (p. 615).
Corollary 1 (p. 614) gives that if τ_1, τ_2 are free distinct involutions, $\tau_1 \circ \tau_2 \neq \tau_2 \circ \tau_1$.
The last assertion can be found in [39] (p. 888). □

Corollary III.9.2
Let K_4 act faithfully on $\mathcal{C}(\mathbb{T})$ via β. Then, $\beta = \gamma^*$ where $K_4 \overset{\gamma}{\curvearrowright} \mathbb{T}$ is faithful, $[\gamma_{(1,0)}, \gamma_{(0,1)}] = I_{\mathbb{T}}$ and either $\gamma_{(1,0)}$ or $\gamma_{(0,1)}$ has exactly two fixed points.

Proof.
Every action β of a locally compact Hausdorff group on an abelian C^*-algebra is the pullback of an action γ on its spectrum, and it is faithful iff its pullback is. Since $\mathcal{C}(\mathbb{T})$ separates the points, $[\gamma_{(1,0)}, \gamma_{(0,1)}] = I_{\mathbb{T}}$ whence at least one of them must not be free, by Proposition III.9.1. Since γ is faithful, this amounts to say that either $\gamma_{(1,0)}$ or $\gamma_{(0,1)}$ has exactly two fixed points. □

Let us gave an instructive example. Consider the C^*-system $(\mathcal{C}(\mathbb{T}), K_4, \beta)$ determined by

$$\begin{cases} (\beta_{(1,0)} f)(z) := f(\overline{z}), & z \in \mathbb{T} \\ (\beta_{(0,1)} f)(z) := f(-\overline{z}), & z \in \mathbb{T} \end{cases}$$

for every $f \in \mathcal{C}(\mathbb{T})$. Clearly, $[\beta_{(1,0)}, \beta_{(0,1)}] = I_{\mathcal{C}(\mathbb{T})}$ and $(\beta_{(1,1)} f)(z) = f(-z)$ $(f \in \mathcal{C}(\mathbb{T}), z \in \mathbb{T})$. In other words, $\beta_{(1,0)} = r^*_{\mathrm{Re}}$, $\beta_{(0,1)} = r^*_{\mathrm{Im}}$ and $\beta_{(1,1)} = R^*_\pi$. Then, $\mathfrak{B}$ spectrally decomposes into

$$\begin{cases} \mathfrak{B}_{(0,0)} = \left\{ f \in \mathcal{C}(\mathbb{T}) \colon \widehat{f}_{2n+1} = 0, \widehat{f}_{2n} = \widehat{f}_{-2n}, n \in \mathbb{Z} \right\} \\ \mathfrak{B}_{(1,1)} = \left\{ f \in \mathcal{C}(\mathbb{T}) \colon \widehat{f}_{2n+1} = 0, \widehat{f}_{2n} = -\widehat{f}_{-2n}, n \in \mathbb{Z} \right\} \\ \mathfrak{B}_{(1,0)} = \left\{ f \in \mathcal{C}(\mathbb{T}) \colon \widehat{f}_{2n} = 0, \widehat{f}_{2n+1} = -\widehat{f}_{-2n-1}, n \in \mathbb{Z} \right\} \\ \mathfrak{B}_{(0,1)} = \left\{ f \in \mathcal{C}(\mathbb{T}) \colon \widehat{f}_{2n} = 0, \widehat{f}_{2n+1} = \widehat{f}_{-2n-1}, n \in \mathbb{Z} \right\} \end{cases}$$

Recall that for every $f \in \mathcal{C}(\mathbb{T})$ and $n \in \mathbb{Z}$, the n-th Fourier coefficient of f is defined as $\widehat{f}_n := \int_{\mathbb{T}} f(z) z^{-n} \mathrm{d}z$, where $\mathrm{d}z$ denotes the probability Haar measure on $\mathbb{T}$. Then,

- $f = \sum_{n \in \mathbb{Z}} \widehat{f}_n z^n$ unconditionally in L^p-norm, with $p \in (1, +\infty)$
- by the Carleson-Hunt theorem (1968), $f = \sum_{n \in \mathbb{Z}} \widehat{f}_n z^n$ Haar almost everywhere
- by the Fejér theorem (1904), $f = \sum_{n \in \mathbb{Z}} \widehat{f}_n z^n$ uniformly in the Cesàro sense:

  $$s_k := \sum_{|j| \leq k} \widehat{f}_j z^j \ (k \in \mathbb{N}_0),\ \sigma_n := \frac{1}{n} \sum_{k=0}^{n-1} s_k \ (n \in \mathbb{N}_0), \text{ then } \|\sigma_n - f\|_{\infty, \mathbb{T}} \xrightarrow{n \uparrow +\infty} 0$$

- by the Stone-Weiestrass theorem (1948), the Wiener algebra

  $$W(\mathbb{T}) := \mathcal{F}^{-1}(\ell^1(\mathbb{Z})) = \{ f \in L^1(\mathbb{T}) \colon (\widehat{f}_n)_{n \in \mathbb{Z}} \in \ell^1(\mathbb{Z}) \}$$

  is a uniformly dense *-subalgebra of $\mathcal{C}(\mathbb{T})$ $(\mathcal{F} \colon L^1(\mathbb{T}) \to c_0(\mathbb{Z}),\ \mathcal{F}(f) := (\widehat{f}_n)_{n \in \mathbb{Z}}$ is the Fourier transform on $L^1(\mathbb{T}))$

In any of the meaning reported above, we can then write

$$\begin{cases} \mathfrak{B}_{(0,0)} = \left\{ f \in \mathcal{C}(\mathbb{T}) \colon f = \widehat{f}_0 + 2\sum_{n\in\mathbb{N}} \widehat{f}_{2n}\mathrm{Re}(z^{2n}) \right\} = \mathcal{C}(\mathbb{T})_{\mathrm{e,r}} \\ \mathfrak{B}_{(1,1)} = \left\{ f \in \mathcal{C}(\mathbb{T}) \colon f = \widehat{f}_0 + 2i\sum_{n\in\mathbb{N}} \widehat{f}_{2n}\mathrm{Im}(z^{2n}) \right\} = \mathcal{C}(\mathbb{T})_{\mathrm{e,i}} \\ \mathfrak{B}_{(1,0)} = \left\{ f \in \mathcal{C}(\mathbb{T}) \colon f = 2i\sum_{n\in\mathbb{N}_0} \widehat{f}_{2n+1}\mathrm{Im}(z^{2n+1}) \right\} = \mathcal{C}(\mathbb{T})_{\mathrm{o,i}} \\ \mathfrak{B}_{(0,1)} = \left\{ f \in \mathcal{C}(\mathbb{T}) \colon f = 2\sum_{n\in\mathbb{N}_0} \widehat{f}_{2n+1}\mathrm{Re}(z^{2n+1}) \right\} = \mathcal{C}(\mathbb{T})_{\mathrm{o,r}} \end{cases}$$

where the subscripts "e", "o", "r", "i" stand for "even", "odd", "real" and "imaginary", respectively. Hence,

$$\mathfrak{B}_{\Delta_+} = \mathcal{C}(\mathbb{T})_{\mathrm{e,r}} \oplus \mathcal{C}(\mathbb{T})_{\mathrm{e,i}} = \mathcal{C}(\mathbb{T})_{\mathrm{e}}$$

is the C^*-algebra of the *even* functions on $\mathbb{T}$, whereas

$$\mathfrak{B}_{\Delta_-} = \mathcal{C}(\mathbb{T})_{\mathrm{o,r}} \oplus \mathcal{C}(\mathbb{T})_{\mathrm{o,i}} = \mathcal{C}(\mathbb{T})_{\mathrm{o}}$$

is the closed subspace of the *odd* ones.

Since $\mathfrak{B} = \mathcal{C}(\mathbb{T})$ is clearly nuclear and separable, we can build its Klein C^*-chain $\mathfrak{A}$ and apply Theorem III.8.10. For this purpose, observe that $\mathcal{S}_{\Delta_+}(\mathfrak{B}) \cong \mathcal{S}(\mathcal{C}(\mathbb{T})_{\mathrm{e}}) \cong \mathcal{M}_1([0,1])$. Then, for each $\varphi \in \mathcal{S}_{\mathfrak{G}}(\mathfrak{A})$, there exists a unique $\prec$-maximal $\mu_\varphi \in \mathcal{M}_1(\mathcal{S}_{\mathfrak{G}}(\mathfrak{A}))$ supported by

$$\mathcal{E}_{\mathfrak{G}}(\mathfrak{A}) = \left\{ \prod_{n\in\mathbb{N}} \psi \right\}_{\psi\in\mathcal{S}_{\Delta_+}(\mathfrak{B})} \cong \left\{ \bigotimes_{n\in\mathbb{N}} \nu \right\}_{\nu\in\mathcal{M}_1([0,1])} \quad \text{s.t. } \varphi = \int_{\mathcal{E}_{\mathfrak{G}}(\mathfrak{A})} \omega \,\mathrm{d}\mu_\varphi(\omega).$$

We conclude this example by observing that any non-trivial automorphism $\gamma \in \mathsf{Aut}(K_4) \cong \mathfrak{S}_{\mathbf{3}}$ produces a new C^*-system $(\widetilde{\mathfrak{B}}, K_4, \widetilde{\beta})$, where $\widetilde{\mathfrak{B}} := \mathcal{C}(\mathbb{T})$ and $\widetilde{\beta}_g := \beta_{\gamma(g)}$, which is conjugate to the original one, by the very definition. The action $\widetilde{\beta}$ induces a K_4-grading on $\widetilde{\mathfrak{B}}$ (where $\widetilde{\mathfrak{B}}_\sigma = \mathfrak{B}_{\sigma\circ\gamma^{-1}}$, $\sigma \in K_4$), with its associated Klein C^*-chain $\widetilde{\mathfrak{A}} := \left(\circledK_{n\in\mathbb{N}} \widetilde{\mathfrak{B}}, K_4, \delta^{(\widetilde{\beta})} \right)$. By identifying the set $\{(1,0),(0,1),(1,1)\}$ with $\mathbf{3} = \{1,2,3\}$, it is easy to see that

- if $\gamma = (1\ 2) \in \mathfrak{S}_{\mathbf{3}}$, then $\widetilde{\mathfrak{A}} \cong \mathfrak{A}$
- $(2\ 3), (1\ 3\ 2) \in \mathfrak{S}_{\mathbf{3}}$ produce isomorphic Klein C^*-chains, where $\widetilde{\mathfrak{B}}_{\Delta_+} = \mathcal{C}(\mathbb{T})_{\mathrm{e,r}} \oplus \mathcal{C}(\mathbb{T})_{\mathrm{o,i}}$
- $(1\ 3), (1\ 2\ 3) \in \mathfrak{S}_{\mathbf{3}}$ produce isomorphic Klein C^*-chains, where $\widetilde{\mathfrak{B}}_{\Delta_+} = \mathcal{C}(\mathbb{T})_{\mathrm{e,r}} \oplus \mathcal{C}(\mathbb{T})_{\mathrm{o,r}}$

III.9.2 Compact operators

Let $\mathcal{H}$ be a (complex) Hilbert space and $\mathfrak{B} := \mathcal{K}(\mathcal{H}) \subseteq \mathcal{B}(\mathcal{H})$ the closed, two-sided *-ideal of compact operators on $\mathcal{H}$. In such a case, $\mathfrak{B}$ is nuclear and simple, and if $\mathcal{H}$ is separable $\mathfrak{B}$ is a separable, maximal ideal in $\mathcal{B}(\mathcal{H})$. It is well-known that $\mathsf{Aut}(\mathfrak{B}) = \{\mathrm{ad}_U \colon U \in \mathcal{U}(\mathcal{H})\}$, whence every action β of a group G (without topological requirements) on $\mathfrak{B}$ is *pointwise* unitarily implemented, i.e. for each $g \in G$ there exists $U(g) \in \mathcal{U}(\mathcal{H})$ s.t. $\beta_g = \mathrm{ad}_{U(g)}$. More generally, for every map $\varrho \colon G \to \mathbb{T}$, $\beta_g = \mathrm{ad}_{\varrho(g)U(g)}$ for each $g \in G$. By irreducibility of $\mathfrak{B}$, the assignment $U_\varrho \colon g \mapsto \varrho(g)U(g)$ defines a *unitary-projective* representation of G on $\mathcal{H}$ i.e. there exists a 2-cocycle/multiplier $\omega_\varrho \in Z^2(G,\mathbb{T})$ s.t. $U_\varrho(g)U_\varrho(g') = \omega_\varrho(g,g')U_\varrho(gg')$ for every $g, g' \in G$ (in particular, $U_\varrho(e) = \omega_\varrho(e,e)I_{\mathcal{H}})$. Observe that if $\varrho,\varrho'\colon G\to\mathbb{T}$, then $U_{\varrho'}(g) = \dfrac{\varrho'(g)}{\varrho(g)}U_\varrho(g)$ $(g\in G)$, which implies

$$\omega_{\varrho'}(g,g') = \frac{\varrho'(g)\varrho'(g')}{\varrho'(gg')}\frac{\varrho(gg')}{\varrho(g)\varrho(g')}\omega_\varrho(g,g'),\quad g,g'\in G\,.$$

In other words, if $\mathcal{PU}(\mathcal{H}) := \mathcal{U}(\mathcal{H})/\mathbb{T}$ is the unitary-projective group on $\mathcal{H}$ and $H^2(G,\mathbb{T})$ is the 2-cohomology class of G, $U_{\varrho'}(g)\sim_{\mathcal{PU}(\mathcal{H})} U_\varrho(g)$ $(g\in G)$ and $\omega_{\varrho'}\sim_{H^2(G,\mathbb{T})}\omega_\varrho$. In particular, there exists a 1-1 correspondence

$$\begin{aligned}\{\text{actions of } G \text{ on } \mathfrak{B}\} &\longleftrightarrow \mathsf{Hom}(G,\mathcal{PU}(\mathcal{H}))\\ \mathrm{ad}_{U(g)} &\longleftrightarrow [g\mapsto [U(g)]_{\mathcal{PU}(\mathcal{H})}]\end{aligned}$$

Evidently, the action of G on $\mathfrak{B}$ is faithful if and only if the associated group homomorphism in $\mathsf{Hom}(G,\mathcal{PU}(\mathcal{H}))$ is injective. If $G=K_4$, then a faithful action $K_4 \overset{\beta}{\curvearrowright}\mathfrak{B}$ is uniquely determined by a pair $(U,V)\in\mathcal{U}(\mathcal{H})\times\mathcal{U}(\mathcal{H})$ s.t. (once named $\sim:=\sim_{\mathcal{PU}(\mathcal{H})}$)

$$\begin{cases} U\not\sim V\not\sim I_{\mathcal{H}}\\ U^2\sim V^2\sim I_{\mathcal{H}}\\ UV\sim VU\end{cases}$$

This is done via $\beta_{(1,0)} := \mathrm{ad}_U$, $\beta_{(0,1)} := \mathrm{ad}_V$, and the spectral decomposition of $\mathfrak{B}$ then becomes

$$\begin{aligned}&\mathfrak{B}_{(0,0)} = \{K\in\mathcal{K}(\mathcal{H})\colon [K,U]=[K,V]=0\} &&\mathfrak{B}_{(1,0)} = \{K\in\mathcal{K}(\mathcal{H})\colon \{K,U\}=[K,V]=0\}\\ &\mathfrak{B}_{(1,1)} = \{K\in\mathcal{K}(\mathcal{H})\colon \{K,U\}=\{K,V\}=0\} &&\mathfrak{B}_{(0,1)} = \{K\in\mathcal{K}(\mathcal{H})\colon [K,U]=\{K,V\}=0\}\end{aligned}$$

Since $H^2(K_4,\mathbb{T}) = \{[\mathbb{1}_{K_4\times K_4}],[\omega]\}\cong\mathbb{Z}_2$ where $\omega(\mathbf{x},\mathbf{y}) := (-1)^{x_2y_1}$, $\mathbf{x},\mathbf{y}\in K_4$, from the previous discussion a unitary-projective representation $(U',\omega)\colon K_4\to\mathcal{U}(\mathcal{H})$ induces an action $\beta\colon g\mapsto\mathrm{ad}_{U'(g)}$ which cannot be implemented by a proper unitary representation. For instance, a modified version of the $\mathbb{Z}_2$-action on the full matrix algebra $\mathfrak{B} := \mathcal{K}(\mathbb{C}^2) = \mathcal{B}(\mathbb{C}^2) = M_2(\mathbb{C})$ considered at the beginning of Subsection III.7.2 gives a faithful action

$$\begin{aligned}\beta\colon K_4 &\to \mathsf{Aut}(M_2(\mathbb{C}))\\ \mathbf{x} &\mapsto \mathrm{ad}_{U^{x_1}V^{x_2}}\end{aligned}$$

where $U := \left[\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right]$, $V := \left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]$, $UV = \left[\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right]$. This action is proposed in Example 3.5 of [113] (p. 24) and it is implemented by the unitary-projective representation $(U',\omega)\colon\mathbf{x}\mapsto U^{x_1}V^{x_2}$, $\mathbf{x}\in K_4$. Notice that $[\beta_{(1,0)},\beta_{(0,1)}]_{\mathsf{Aut}(M_2(\mathbb{C}))} = [\mathrm{ad}_U,\mathrm{ad}_V]_{\mathsf{Aut}(M_2(\mathbb{C}))} = I_{M_2(\mathbb{C})}$ and $VU=-UV$. Interestingly, it can be also shown that $K_4\ltimes_{\beta,\mathrm{f}} M_2(\mathbb{C})\cong K_4\ltimes_{\beta,\mathrm{r}} M_2(\mathbb{C})\cong M_4(\mathbb{C})$. We have

$$\begin{aligned}&\mathfrak{B}_{(0,0)} = \mathbb{C}I_2 &&\mathfrak{B}_{(1,0)} = \mathbb{C}V\\ &\mathfrak{B}_{(1,1)} = \mathbb{C}UV &&\mathfrak{B}_{(0,1)} = \mathbb{C}U\end{aligned}$$

so that $\mathfrak{B}_{\Delta_+} = \left\{\left[\begin{smallmatrix}a&b\\-b&a\end{smallmatrix}\right]\colon a,b\in\mathbb{C}\right\}\cong\mathbb{C}^2$ and $\mathcal{S}_{\Delta_+}(\mathfrak{B})\cong\mathcal{M}_1(\{0,1\})\cong[0,1]$. Again, $\mathfrak{B}=M_2(\mathbb{C})$ is nuclear and separable, whence Theorem III.8.10 is perfectly applicable. In particular, $\mathcal{E}_{\mathfrak{S}}(\mathfrak{A})\cong[0,1]^{\mathbb{N}}$.

III.9.3 Irrational rotation algebras

P. J. Stacey devotes a paper to the explicit construction of a faithful K_4-action on the irrational rotation algebra $A_\vartheta := C^*_{\mathrm{u}}(u,v\,|\,vu = e^{2\pi i\vartheta}uv)$, with $\vartheta\in(0,1)$ irrational (see [74]). To accomplish that, a description of A_ϑ as C^*-inductive limit of a suitable direct system (due to

Elliott and Evans in [25], Section 5, p. 498) is exploited.

Let $[m_0; m_1, m_2, m_3, \ldots]$ be the (infinite) regular continued fraction representation of ϑ, where $m_0 = 0$, and $\dfrac{p_n}{q_n} := [m_0; m_1, \ldots, m_n] \in \mathbb{Q}$ the n-th *convergent* of ϑ, for $n \geq 1$. Also, let $a_n, b_n, c_n, d_n \in \mathbb{N}$ be obtained by

$$\begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix} := \prod_{k=0}^{3} \begin{bmatrix} m_{4n-k} & 1 \\ 1 & 0 \end{bmatrix} \in SL_2(\mathbb{Z}), \quad n \geq 1$$

and consider the unitary $k \times k$ matrices over $\mathcal{C}(\mathbb{T})$ given by

$$R_k := \begin{bmatrix} 0 & \mathrm{id}_{\mathbb{T}} \\ I_{k-1} & 0 \end{bmatrix}, \quad S_k := \begin{bmatrix} 0 & \mathbb{1}_{\mathbb{T}} \\ I_{k-1} & 0 \end{bmatrix} \in U_k(\mathbb{C}) \otimes \mathcal{C}(\mathbb{T}) \cong U_k(\mathcal{C}(\mathbb{T})), \quad k \geq 1$$

Then, by Theorem 4 in [25] (p. 497), $A_\vartheta \cong \overline{\varinjlim_n (A_n, \phi_{nm})_{n \leq m}}^{C^*}$, where $A_n := M_{q_{4n}}(\mathcal{C}(\mathbb{T})) \oplus M_{q_{4n-1}}(\mathcal{C}(\mathbb{T}))$ $(n \geq 1)$ and the adjacent connecting maps are defined through blocks of (non-commutative) Kronecker products of matrices as

$$\begin{aligned} \phi_{n,n+1} \colon A_n &\to A_{n+1} \\ (\mathrm{id}_{\mathbb{T}} I_{q_{4n}}, 0) &\mapsto \left(\begin{bmatrix} I_{q_{4n}} \otimes R_{a_{n+1}} & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} I_{q_{4n}} \otimes S_{c_{n+1}} & 0 \\ 0 & 0 \end{bmatrix} \right) \\ (0, \mathrm{id}_{\mathbb{T}} I_{q_{4n-1}}) &\mapsto \left(\begin{bmatrix} 0 & 0 \\ 0 & I_{q_{4n-1}} \otimes S_{b_{n+1}} \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & I_{q_{4n-1}} \otimes R_{d_{n+1}} \end{bmatrix} \right) \\ (X, Y) &\mapsto \left(\begin{bmatrix} X \otimes I_{a_{n+1}} & 0 \\ 0 & Y \otimes I_{b_{n+1}} \end{bmatrix}, \begin{bmatrix} X \otimes I_{c_{n+1}} & 0 \\ 0 & Y \otimes I_{d_{n+1}} \end{bmatrix} \right) \end{aligned}$$

for each $n \geq 1$. Now, $K_0(A_n) \cong K_1(A_n) \cong \mathbb{Z} \times \mathbb{Z}$ for every $n \geq 1$ and, under identifications of these K-groups with $\mathbb{Z} \times \mathbb{Z}$, the connecting maps $\phi_{n,n+1}$ induce group automorphisms at the K_0 and K_1-levels given by $(\phi_{n,n+1})_0 = \begin{bmatrix} a_{n+1} & b_{n+1} \\ c_{n+1} & d_{n+1} \end{bmatrix}$ and $(\phi_{n,n+1})_1 = I_2$, respectively. At this stage, Proposition 1.2 (p. 137) and Theorem 1.1 (p. 138) in [74] give the following results, where $I_k := \mathrm{diag}(1 \ldots 1)$ and $J_k := \mathrm{antidiag}(1 \ldots 1)$, $k \geq 1$.

Theorem III.9.3 (Stacey, 1997)
Let $(W_k)_{k \geq 1}, (W'_k)_{k \geq 1}, (V_k)_{k \geq 1}, (V'_k)_{k \geq 1}$ be four families of unitary, involutive matrices recursively defined by

$$\begin{cases} W_1 := I_{q_4} \\ V_1 := J_{q_3} \\ W_{n+1} := \begin{bmatrix} W_n \otimes I_{a_{n+1}} & 0 \\ 0 & V_n \otimes J_{b_{n+1}} \end{bmatrix}, n \geq 1 \\ V_{n+1} := \begin{bmatrix} W_n \otimes I_{c_{n+1}} & 0 \\ 0 & V_n \otimes J_{d_{n+1}} \end{bmatrix}, n \geq 1 \end{cases} \qquad \begin{cases} W'_1 := J_{q_4} \\ V'_1 := I_{q_3} \\ W'_{n+1} := \begin{bmatrix} W'_n \otimes J_{a_{n+1}} & 0 \\ 0 & V'_n \otimes I_{b_{n+1}} \end{bmatrix}, n \geq 1 \\ V'_{n+1} := \begin{bmatrix} W'_n \otimes J_{c_{n+1}} & 0 \\ 0 & V'_n \otimes I_{d_{n+1}} \end{bmatrix}, n \geq 1 \end{cases}$$

By considering $W_n, W'_n \in \mathcal{C}(\mathbb{T}, M_{q_{4n}})$, $V_n, V'_n \in \mathcal{C}(\mathbb{T}, M_{q_{4n-1}})$ as constant selfadjoint functions for every $n \geq 1$, $\Sigma_n := \mathrm{ad}_{W_n} \oplus (\mathrm{ad}_{V_n} \circ r_{\mathrm{Re}})$ and $\Sigma'_n := (\mathrm{ad}_{W'_n} \circ r_{\mathrm{Re}}) \oplus \mathrm{ad}_{V'_n}$

- are involutive, commuting automorphisms of $\mathcal{C}(\mathbb{T}, M_{q_{4n}}(\mathbb{C})) \oplus \mathcal{C}(\mathbb{T}, M_{q_{4n-1}}(\mathbb{C})) \cong A_n$
- $(\Sigma)_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \in GL_2(\mathbb{Z}) \cong \mathsf{Aut}(K_1(A_n))$ and $(\Sigma')_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \in GL_2(\mathbb{Z}) \cong \mathsf{Aut}(K_1(A_n))$

In particular, there exists a faithful action $K_4 \overset{\beta}{\curvearrowright} A_\vartheta$ s.t. $(\beta_{(1,0)})_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \in GL_2(\mathbb{Z}) \cong \mathsf{Aut}(K_1(A_\vartheta))$ and $(\beta_{(0,1)})_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \in GL_2(\mathbb{Z}) \cong \mathsf{Aut}(K_1(A_\vartheta))$.

In Section 2 of [74], Stacey investigates the unital C^*-subalgebra $\mathrm{Fix}(\beta_{(1,1)}) = (A_\vartheta)_{(0,0)} \oplus (A_\vartheta)_{(1,1)}$, showing that it is isomorphic to the fixed point algebra of another C^*-system $(A_\vartheta, \mathbb{Z}_2, \sigma)$, where σ_1 is the unique *-endomorphism extending the involutive map $u \mapsto u^*$, $v \mapsto v^*$ (existence and unicity of σ_1 are guaranteed by the universal property of A_ϑ). It results that σ_1 is involutive as well, hence $\sigma_1 \in \mathsf{Aut}(A_\vartheta)$; we will denote it simply by σ from now on. The associated fixed point algebra $A_\vartheta^\sigma = C^*(u + u^*, v + v^*) \subset A_\vartheta$, also called *symmetrized* non-commutative torus,

(1) has K-theory $K_0(A_\vartheta^\sigma) \cong \mathbb{Z}^6$ and $K_1(A_\vartheta^\sigma) \cong (0)$ ([53])

(2) is simple (Theorem 8.10.12, p. 445 in [97])

(3) is approximately finite-dimensional (Theorem 1.1, p. 606 in [12])

(4) has a unique tracial state, coinciding with the restriction of that on A_ϑ (Theorem 4.5, p. 162 in [11]). In particular, it is faithful.

Theorem 3.6 (p. 157) in [11] gives an explicit presentation of A_ϑ^σ, though not so elegant:

$$A_\vartheta^\sigma \cong C^*\left(s, t \,|\, s = s^*, t = t^*, R_1, R_2, R_3\right)$$

where

R_1) $s^2 \bullet t = \cos(2\pi\vartheta) sts + 2\sin^2(2\pi\vartheta) t$

R_2) $s \bullet t^2 = \cos(2\pi\vartheta) tst + 2\sin^2(2\pi\vartheta) s$

R_3) $i\,\mathrm{Im}\left((ts)^2\right) = \cos(4\pi\vartheta)(st)^2 - \cos(2\pi\vartheta) s^2 t^2 + (\cos(2\pi\vartheta) - \cos(6\pi\vartheta))(s^2 + t^2 - \mathbb{1})$

To conclude, since $\mathfrak{B} := A_\vartheta$ is nuclear and separable, once more we can apply Theorem III.8.10 to its Klein C^*-chain $\mathfrak{A}$. Given the above action $K_4 \overset{\beta}{\curvearrowright} \mathfrak{B}$, $\mathfrak{B}_{\Delta_+} \cong A_\vartheta^\sigma = C^*(u + u^*, v + v^*)$. Therefore, for every $\varphi \in \mathcal{S}_{\mathfrak{G}}(\mathfrak{A})$, there is a unique $\prec$-maximal $\mu_\varphi \in \mathcal{M}_1(\mathcal{S}_{\mathfrak{G}}(\mathfrak{A}))$ supported by $\mathcal{E}_{\mathfrak{G}}(\mathfrak{A}) \cong \left\{ \prod_{n \in \mathbb{N}} \psi \right\}_{\psi \in \mathcal{S}(A_\vartheta^\sigma)}$ satisfying $\varphi = \int_{\mathcal{E}_{\mathfrak{G}}(\mathfrak{A})} \omega \,\mathrm{d}\mu_\varphi(\omega)$.

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